

RICE UNIVERSITY

**Higher-Order Analogues of Genus and
Slice Genus of Classical Knots**

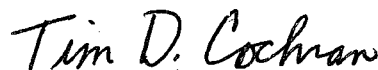
by

Peter Douglas Horn

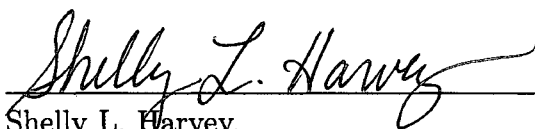
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Abstract

Higher-Order Analogues of Genus and

Slice Genus of Classical Knots

by

Peter Douglas Horn

We define invariants analogous to the genus and slice genus of knots in S^3 . For algebraically slice, genus one knots, we define the differential genus, denoted dg , and we prove it is independent of the Alexander polynomial and knot Floer homology. For knots which bound Grope of height $n + 2$ in D^4 , we define the n^{th} -order genus, denoted g_n . Each of the n^{th} -order genera is a generalization of the slice genus. For each $n \geq 1$, we construct knots with identical lower-order genera and distinct n^{th} -order genera, thus proving that these invariants are independent of one another. Finally, we employ the higher-order genera to give a refinement of the Grope filtration of the knot concordance group.

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Chapter 1

Introduction

1.1 Background

A **knot** is a smooth embedding of the circle into the three-sphere, $K : S^1 \hookrightarrow S^3$. Two knots K_0 and K_1 are isotopic if there is an isotopy $\widehat{K} : S^3 \times [0, 1] \rightarrow S^3$ such that $\widehat{K}(-, 0) = K_0$ and $\widehat{K}(-, 1) = K_1$. By a “knot,” we usually mean the isotopy class of a smooth embedding $S^1 \hookrightarrow S^3$. Occasionally, it will be necessary to orient a knot. An orientation on a knot is induced by a choice of orientation on the circle.

Consider $S^3 \times [0, 1]$. An oriented knot $K_0 \subset S^3 \times \{0\}$ is **topologically concordant** to an oriented knot $K_1 \subset S^3 \times \{1\}$ if there is an oriented, topologically flat annulus $A \subset S^3 \times [0, 1]$ such that $A \cap S^3 \times \{i\} = K_i$ for $i = 0, 1$. Oftentimes we will omit the word “oriented” when we talk about concordance, as an orientation on the annulus will determine the orientations on the boundary knots. An oriented knot $K_0 \subset S^3 \times \{0\}$ is **smoothly concordant** to an oriented knot $K_1 \subset S^3 \times \{1\}$ if there is an oriented, smooth annulus $A \subset S^3 \times [0, 1]$ such that $A \cap S^3 \times \{i\} = K_i$ for $i = 0, 1$. Smooth

concordance implies topological concordance, and counterexamples are known to the converse. Concordance is a four-dimensional notion of knot equivalence, and isotopic knots must be concordant.

A knot invariant is a function from the set of knots modulo isotopy into some set. Similarly, a concordance invariant is a function from the set of oriented knots modulo concordance into some set. In this thesis, we will define higher-order analogues of the knot invariant called the “genus” of a knot and the concordance invariant called the “slice genus” of a knot.

The genus of a knot is defined as follows. Given an oriented diagram of a knot K , Seifert’s algorithm [Sei] produces a compact, oriented surface Σ in S^3 with $\partial\Sigma = K$. Such a surface is called a **Seifert surface** for K . The **genus** of K , denoted $g(K)$, is the smallest genus of all Seifert surfaces for K . If two knots are isotopic, the isotopy between them carries Seifert surfaces for one knot to Seifert surfaces for the other. Thus, the genus of a knot is an invariant of knot type. Knot genus is a knot invariant that takes values in the non-negative integers. The only knot with genus equal to zero is the unknot. For emphasis, we will occasionally refer to the genus of a knot as the three-genus.

View S^3 as the boundary of D^4 . Pushing a Seifert surface for a knot into D^4 , one sees that every knot bounds a compact, oriented surface that is properly embedded in D^4 . There are two types of such embeddings that we will consider: topologically flat embeddings and smooth embeddings. The **topological slice genus** of K is the minimal genus of all topologically flat, compact, oriented surfaces properly embedded in D^4 with boundary K . Similarly, the **smooth slice genus** of K is the minimal genus

of all smooth, compact, oriented surfaces properly embedded in D^4 with boundary K . These invariants (although we have not yet proven they are concordance invariants) will be denoted $g^t(K)$ and $g^s(K)$, respectively. Let K and J be concordant via an annulus A , and let Σ be a surface in D^4 bounded by J . To see that the (topological or smooth) slice genus is a concordance invariant, one must observe that gluing the annulus A to the surface Σ yields a surface bounded by K that has the same genus as Σ . The slice genus of a knot is a concordance invariant that takes values in the non-negative integers. Any knot that has (topological/smooth) slice genus equal to zero is called a **(topologically/smoothly) slice knot**. One may show that K is (topologically/smoothly) slice if and only if K is (topologically/smoothly) concordant to the unknot.

The set of oriented knots modulo concordance with the connected sum operation forms an abelian group. This group is called the **knot concordance group**, denoted \mathcal{C} (there are actually two groups, the topological and the smooth concordance groups).

1.1.1 Higher-order knot theory

Let $N(K)$ denote a tubular neighborhood of a knot K , and let $Y = S^3 - \text{int}(N(K))$. One can show that $H_1(Y) \cong \mathbb{Z}$. The abelianization map $\pi_1(Y) \rightarrow \mathbb{Z}$ induces a covering space $Y_1 \rightarrow Y$. The covering space Y_1 is called the universal abelian cover of Y or the infinite cyclic cover of Y . The “classical” knot invariants can be defined using the spaces Y and Y_1 . For example, the genus of a knot is a classical invariant. The Alexander polynomial, which is derived from the first homology of the covering

space Y_1 , is also a classical invariant.

A “higher-order” knot invariant is defined using a higher cover of Y . For example, let Y_2 denote the universal abelian cover of Y_1 . Any invariants of K derived from Y_2 would be called higher-order invariants of K .

Higher-order knot concordance also involves covering spaces. T. Cochran, K. Orr, and P. Teichner introduced two filtrations of the concordance group [COT1]. We will discuss these filtrations in more depth in Sections 1.1.2 and 1.1.3. The (n) -solvable filtration is indexed by $\frac{1}{2}\mathbb{N} \cup \{0\}$

$$\{\text{slice knots}\} \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

The classical concordance invariants include the slice genus and the signature and are defined for all knots in \mathcal{C} . The n^{th} -order invariants may only be defined for knots lying in \mathcal{F}_n , and they provide obstructions for a knot to lie in $\mathcal{F}_{n.5}$. The higher-order invariants have been used to reveal some of the rich structure of the knot concordance group [COT1], [COT2], [CT], [CHL4].

Many of the previous higher-order invariants are algebraic. Cochran, Orr, and Teichner’s invariants are L^2 -signatures [COT1]. Subsequently, S. Harvey defined a series of signature invariants to study the string link concordance group [Har]. Using a series of covering spaces, Cochran defined the higher-order Alexander polynomials [Coc], and C. Leidy defined the higher-order Blanchfield pairing [Lei]. More recently, Cochran, Harvey, and Leidy successfully used L^2 -signatures to show that (half of) the quotients of successive terms in the (n) -solvable filtration, $\mathcal{F}_n/\mathcal{F}_{n.5}$, have infinite rank [CHL4]. These invariants generalize the Alexander polynomial and classical

signature.

The author has defined the first higher-order geometric invariant of knots [Hor2]. In this thesis, we discuss a higher-order analogue of the three-genus, and we provide a series of higher-order analogues of the slice genus.

1.1.2 The (n) -solvable filtration

The (n) -solvable filtration

$$\cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

is defined in terms of algebraic properties on the second homology of certain 4-manifolds, whose boundary is 0-surgery on a knot.

If G is a group, the **derived series of G** is defined recursively by setting $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. The **rational derived series of G** is defined recursively by setting $G_r^{(0)} = G$ and $G_r^{(i+1)} = \{g \in G : g^k \in [G_r^{(i)}, G_r^{(i)}], \text{ for some } k > 0\}$.

Definition 1.1. [COT1] Let M be closed, orientable 3-manifold. A spin 4-manifold W with $\partial W = M$ is an (n) -**solution for M** if the inclusion-induced map $i_* : H_1(M) \rightarrow H_1(W)$ is an isomorphism and if there are embedded surfaces L_i and D_i (with product neighborhoods) for $i = 1, \dots, m$ that satisfy the following conditions:

1. the L_i are disjoint,
2. the D_i are disjoint,

3. the homology classes $\{[L_1], [D_1], \dots, [L_m], [D_m]\}$ form an ordered basis for $H_2(W)$,
4. the intersection form $(H_2(W), \cdot)$ with respect to this ordered basis is a direct sum of hyperbolics,
5. $L_i \cap D_j$ is empty if $i \neq j$,
6. for each i , L_i and D_i intersect transversely at one point, and
7. each L_i and D_i are (n) -surfaces, i.e. $\pi_1(L_i) \subset \pi_1(W)^{(n)}$ and $\pi_1(D_i) \subset \pi_1(W)^{(n)}$.

If, in addition, $\pi_1(L_i) \subset \pi_1(W)^{(n+1)}$ for each i , we say W is an $(n.5)$ -**solution** for M .

If a closed, orientable 3-manifold has an (n) -solution, we say M is (n) -**solvable**.

A knot K in S^3 is an (n) -**solvable knot** if the zero surgery on K is (n) -solvable. A slice knot is (n) -solvable for all $n \in \frac{1}{2}\mathbb{N}$.

As in [COT1], the set of all (n) -solvable knots is denoted \mathcal{F}_n , and Cochran, Orr, and Teichner showed that the \mathcal{F}_n form a nested series of subgroups of \mathcal{C} . This series of subgroups is the (n) -**solvable filtration of the knot concordance group**.

1.1.3 The Grope filtration

Definition 1.2. [FT] A **grope** is a special pair (2-complex, base circle). A grope has a **height** $n \in \frac{1}{2}\mathbb{N}$. A grope of height 1 is precisely a compact, oriented surface Σ with a single boundary component (the base circle). For $n \in \mathbb{N}$, a grope of height $n + 1$ is defined recursively as follows: let $\{\alpha_i, \beta_i : i = 1, \dots, g\}$ be a symplectic basis

of curves for Σ , the **first stage of the grope**. Then a **grope of height $n + 1$** is formed by attaching gropes of height n to each α_i and β_i along the base circles. See Figure 1.1.

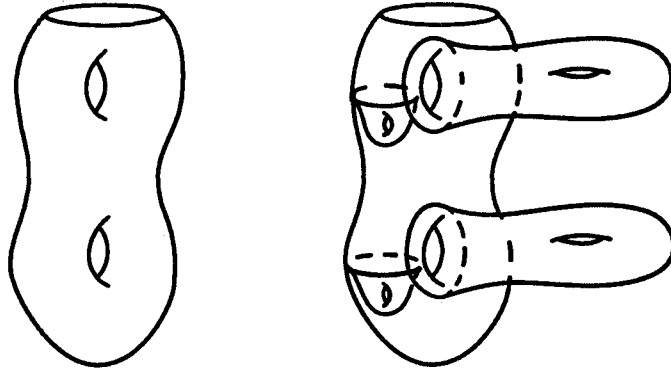


Figure 1.1: A grope of height 1 (a surface) and a grope of height 2

A grope of height 1.5 is formed by attaching gropes of height 1 (i.e. surfaces) to a Lagrangian of a symplectic basis of curves for Σ . That is, a grope of height 1.5 is a surface with surfaces glued to “half” of the basis curves. In general, a grope of height $n + 1.5$ is obtained by attaching gropes of height n to the α_i and gropes of height $n + 1$ to the β_i .

Given a 4-manifold W with boundary M and a framed circle $\gamma \subset M$, we say that γ **bounds a Grope in W** if γ extends to an embedding of a grope with its untwisted framing. That is, a Grope has a trivial normal bundle, so parallel push-offs can be taken. Knots in S^3 are always equipped with the zero framing.

The set of all (concordance classes of) knots that bound Gropes of height n in D^4 is denoted \mathcal{G}_n , which is a subgroup of \mathcal{C} . We may choose to forget the top stages

of a Grope. Thus, if K bounds a Grope of height $n + 1$ in D^4 , K also bounds a Grope of height n in D^4 . We see that $\mathcal{G}_{n+1} \subset \mathcal{G}_n$ as subgroups of \mathcal{C} , and this series of subgroups is **the Grope filtration of the knot concordance group**. The Grope filtration is related to the (n) -solvable filtration in the sense that $\mathcal{G}_{n+2} \subset \mathcal{F}_n$ for all $n \in \frac{1}{2}\mathbb{N} \cup \{0\}$ [COT1, Theorem 8.11].

1.1.4 Knot Floer homology

Knot Floer homology was defined by Peter Ozsváth and Zoltán Szabó [OS1] and by Jacob Rasmussen [Ras]. We are concerned with the “hat version” of the theory. The knot Floer homology of a knot K in S^3 is a bigraded abelian group

$$\widehat{HFK}(K) = \bigoplus_{i,j \in \mathbb{Z}} \widehat{HFK}_j(K, i)$$

The i grading is called the Alexander grading, and the j grading is the Maslov grading. We will not present the definition of knot Floer homology. Knot Floer homology is an extremely powerful knot invariant. For example, the largest i such that $\bigoplus_{j \in \mathbb{Z}} \widehat{HFK}_j(K, i) \neq 0$ is equal to the genus of K [OS2, Theorem 1.2]. In addition, the graded Euler characteristic of knot Floer homology is equal to the knot’s Alexander polynomial [OS1][Ras], that is

$$\sum_{i,j} (-1)^j \text{rank}_{\mathbb{Z}} \widehat{HFK}_j(K, i) \cdot t^i = \Delta_K(t)$$

Thus, the knot Floer homology detects both the genus and Alexander polynomial of a knot, two of the classical knot invariants. Knot Floer homology also yields some information about the (smooth) slice genus. We will investigate whether knot Floer

homology contains any information about our higher-order analogues of the genus of a knot.

1.1.5 L^2 -invariants

L^2 -invariants have many applications in low-dimensional topology. Cochran, Orr, and Teichner pioneered the use of L^2 -invariants in the study of knot concordance [COT1][COT2]. Their initial work inspired further use of L^2 -signatures in knot concordance by T. Cochran, S. Harvey, and C. Leidy [CHL2][CHL4], J. C. Cha [Cha], S. Friedl [Fri], and T. Kim [Kim1][Kim2]. In Chapters 2 and 4, we employ L^2 -invariants in obtaining bounds on our higher-order analogues of genus and slice genus of knots. The definitions in this section can be found in W. Lück’s book on L^2 -invariants [Lüc] and in H. Reich’s dissertation [Rei].

As a motivating example, consider a finite CW complex X and a regular covering space $\tilde{X} \rightarrow X$ whose deck group Γ is discrete. The group ring $\mathbb{Z}\Gamma$ acts on the chain complex $C_*\left(\tilde{X}\right)$, and one can form the twisted chain complex $C_*\left(\tilde{X}; \mathbb{Z}\Gamma\right)$ of right $\mathbb{Z}\Gamma$ -modules. In this setting, one could define “twisted Betti numbers” using $\text{rank}_{\mathbb{Z}\Gamma}$. Unfortunately, $\text{rank}_{\mathbb{Z}\Gamma}$ does not behave well (for instance, it is often infinite). In L^2 -homology, one passes to the group von Neumann algebra $\mathcal{N}\Gamma$ and then defines $\dim_{\mathcal{N}\Gamma}$, which behaves better than $\text{rank}_{\mathbb{Z}\Gamma}$.

Consider the Hilbert space $\ell^2(\Gamma) = \left\{ \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma : \lambda_\gamma \in \mathbb{C}, \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \overline{\lambda_\gamma} < \infty \right\}$ with inner product

$$\left\langle \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma, \sum_{\gamma \in \Gamma} \mu_\gamma \cdot \gamma \right\rangle = \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \overline{\mu_\gamma}$$

Left multiplication by elements of Γ on $\ell^2(\Gamma)$ induces an isometric Γ -action on $\ell^2(\Gamma)$. The group von Neumann algebra $\mathcal{N}\Gamma$ is the C^* -algebra of left Γ -equivariant bounded linear operators on the Hilbert space $\ell^2(\Gamma)$. An important tool in defining $\dim_{\mathcal{N}\Gamma}$ is the von Neumann trace $\mathrm{tr}_{\mathcal{N}\Gamma} : \mathcal{N}\Gamma \rightarrow \mathbb{C}$, defined by $\mathrm{tr}_{\mathcal{N}\Gamma}(f) = \langle f(e), e \rangle$, where $e \in \Gamma \subset \ell^2(\Gamma)$ is the identity element. A Hilbert $\mathcal{N}\Gamma$ -module V is a Hilbert space with a Γ -action such that $V \hookrightarrow \mathcal{H} \otimes_{\mathbb{C}} \ell^2(\Gamma)$ for some separable Hilbert space \mathcal{H} . The von Neumann dimension of a Hilbert $\mathcal{N}\Gamma$ -module V is defined to be

$$\dim_{\mathcal{N}\Gamma} V = \mathrm{tr}_{\mathcal{N}\Gamma}(\mathrm{id} : V \rightarrow V)$$

which takes values in $[0, \infty]$.

Given the singular chain complex $C_*(X)$, one defines the singular homology of X with twisted coefficients in $\mathcal{N}\Gamma$ as

$$H_*(X; \mathcal{N}\Gamma) := H_*(C_*(X) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}\Gamma)$$

Each twisted homology group $H_p(X; \mathcal{N}\Gamma)$ is a right Hilbert $\mathcal{N}\Gamma$ -module, so $\dim_{\mathcal{N}\Gamma} H_p(X; \mathcal{N}\Gamma)$ is defined (this number is the p^{th} L^2 -beti number of X). Let $\mathcal{U}\Gamma$ denote the algebra of operators affiliated to the group von Neumann algebra $\mathcal{N}\Gamma$, that is, all closed, densely defined operators which commute with the action of Γ on $\ell^2(\Gamma)$. We have $\Gamma \hookrightarrow \mathbb{Z}\Gamma \hookrightarrow \mathcal{N}\Gamma \hookrightarrow \mathcal{U}\Gamma$. In particular, we may view $\mathcal{U}\Gamma$ as a $\mathbb{Z}\Gamma$ -module or a Hilbert $\mathcal{N}\Gamma$ -module. The singular homology of X with twisted coefficients in $\mathcal{U}\Gamma$ is defined to be

$$H_*(X; \mathcal{U}\Gamma) := H_*(C_*(X) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma)$$

We are interested in the special case that X is an oriented 4-manifold, Γ a poly-torsion-free-abelian group, and $\varphi : \pi_1(X) \twoheadrightarrow \Gamma$. A group is poly-torsion-free-abelian

(PTFA) if it has a finite subnormal series whose successive quotients are torsion-free abelian groups. For example, if G is any group, the quotient $G/G_r^{(n)}$ is PTFA. The intersection form

$$\lambda : H_2(X; \mathcal{U}\Gamma) \times H_2(X; \mathcal{U}\Gamma) \rightarrow \mathcal{U}\Gamma$$

is a hermitian form [COT1], and so the Hilbert $\mathcal{N}\Gamma$ -module $H_2(X; \mathcal{U}\Gamma)$ decomposes as

$$H_2(X; \mathcal{U}\Gamma) = H_0 \oplus H_+ \oplus H_-$$

where λ is positive-definite on H_+ , negative-definite on H_- , and trivial on H_0 . Each of these submodules is a right Hilbert $\mathcal{N}\Gamma$ -module.

Definition 1.3. The L^2 -signature of X associated to the epimorphism

$\varphi : \pi_1(X) \twoheadrightarrow \Gamma$ is the real number given by

$$\sigma^{(2)}(X, \varphi) := \dim_{\mathcal{N}\Gamma} H_+ - \dim_{\mathcal{N}\Gamma} H_-$$

1.2 Summary of results

For any algebraically slice knot with genus one, we define a 3-dimensional invariant called the differential genus, which is a higher-order analogue of the three-genus. This invariant is denoted dg . We recall the definition of metabelian L^2 -signatures of Cochran, Harvey, and Leidy [CHL2] and prove an analogue of the classical genus-signature inequality of Murasugi [Mur].

Proposition 2.14. *If K is an algebraically slice knot with genus one and non-trivial Alexander polynomial, and if ρ is a metabelian L^2 -signature of K , then $|\rho| \leq 2 \text{dg}(K)$.*

One interesting feature of the differential genus is that it is independent of the classical invariants. Recall that knot Floer homology detects these classical invariants. We would like to address the question of whether knot Floer homology gives any higher-order information about knots. We prove that knot Floer homology does not contain the higher-order information present in the differential genus.

Theorem 2.8. *There is an infinite family of knots with isomorphic knot Floer homology, and any two members of this family can be distinguished by the differential genus.*

To make this more concrete, consider an essential, simple closed curve J on a Seifert surface for K . The curve J lifts to the infinite cyclic cover Y_1 of $Y = S^3 - N(K)$. In particular, J is a non-trivial element of $H_1(Y_1)$. In this sense, information about J gives higher-order information about K . We prove that knot Floer homology does not detect the knottedness of certain curves on Seifert surfaces.

Theorem 2.1. *The knot Floer homology of K does not detect the knottedness of untwisted bands in Seifert surfaces for K .*

We also define higher-order analogues of the slice genus of knots, both in the topological and smooth categories. If the concordance class of a knot K lies in \mathcal{G}_{n+2} , we use the height $n + 2$ Groves bounded by K to define the n^{th} -order genus of K , denoted $g_n(K)$. The n^{th} -order genus is a concordance invariant and satisfies $g_n(K) \geq g^t(K)$. For $K \in \mathcal{G}_{n+2}$, the lower-order genera are also defined, and they satisfy $g_n(K) \geq g_{n-1}(K) \geq \cdots \geq g_0(K) \geq g^t(K)$. Furthermore, $g_n(K) = 0$ if and only if K is a slice knot. We also define the higher-order signatures of knots, which

are certain L^2 -signatures. Analogous to Murasugi's signature-genus inequality [Mur], we prove that one of the n^{th} -order signatures provides a lower bound for the n^{th} -order genus.

Theorem 4.5. *If $K \in \mathcal{G}_{n+2}$, then some n^{th} -order signature σ of K satisfies $|\sigma| \leq 4g_n(K)$.*

With this tool, we are able to prove that the n^{th} -order genus may be arbitrarily larger than the slice genus.

Theorem 4.10. *Let $n \geq 1$, and let C be any positive real number. There exists a constant h_n , depending only on n , and a knot $K \in \mathcal{G}_{n+2}$ with $g^t(K) \leq h_n$ and $C < g_n(K)$.*

We see an immediate corollary.

Corollary 4.12. *There exist infinitely many knots in \mathcal{G}_{n+2} with the same slice genus and distinct n^{th} -order genera.*

Thus, each of the higher-order genera gives more information than the slice genus. With a bit more care, we are able to show that the higher-order genera give more information than the lower-order genera.

Theorem 5.4. *For $n \geq 1$, there is a knot \mathcal{G}_{n+2} whose n^{th} -order genus is arbitrarily larger than all of its lower-order genera (up to order $n - 1$).*

These results show that the higher-order genera provide a geometric refinement of the Grope filtration of the knot concordance group. If one wishes to distinguish two

knots up to concordance, one might first ask how deep in the Grope filtration does each knot lie, and secondly, whether any of the higher-order genera distinguish the two knots.

1.3 Outline of thesis

In Chapter 2, we define the differential genus of algebraically slice, genus one knots. We compute the differential genus for many examples, and we conclude that knot Floer homology does not detect the differential genus.

In Chapter 3, we define the n^{th} -order signatures of (n) -solvable knots, and we show how to construct (n) -solvable knots with the property that each of the n^{th} -order signatures is large.

In Chapter 4, we define the n^{th} -order genus of a knot that lies in \mathcal{G}_{n+2} . We prove that for each such knot, one of its n^{th} -order signatures bounds the n^{th} -order genus from below. As a corollary, we prove that the n^{th} -order genus is independent of the slice genus.

In Chapter 5, we prove that the n^{th} -order genus is independent of each of the lower-order genera, and we explain how these invariants refine the Grope filtration of the knot concordance group.

Chapter 2

The differential genus of a knot

Knot Floer homology was defined by Peter Ozsváth and Zoltán Szabó [OS1] and by Jacob Rasmussen [Ras]. Knot Floer homology is a powerful knot invariant, and it detects such information as the Alexander polynomial [OS1][Ras] and knot genus [OS2, Theorem 1.2]. Either of these invariants can be computed from a minimal genus Seifert surface. We investigate whether knot Floer homology contains more information about any minimal genus Seifert surface.

In [Hor2], the author defined a geometric invariant for knots in S^3 called the **first-order genus**. Roughly, the first-order genus of K is obtained by adding the individual genera (in $S^3 - K$) of the curves in a symplectic basis on a minimal genus Seifert surface for K , and taking the minimum over all minimal genus Seifert surfaces. The first-order genus of a knot is difficult to compute, as there are many symplectic bases for a given Seifert surface. While difficult to compute in general, the first-order genus is a notion of higher-order genus defined for all knots.

Here we define a similar invariant, though it is only defined for algebraically slice,

genus one knots. We take a minimum over Seifert surfaces, but what we record is the genus (in $S^3 - K$) of a basis curve which inherits the zero framing from the surface. We will provide many examples and show that this invariant is not detected by knot Floer homology.

Theorem 2.1. *Knot Floer homology does not detect the knottedness of untwisted bands in a Seifert surface.*

Cochran, Harvey, and Leidy [CHL2] defined the first-order L^2 -signatures of a knot. By their definition, each algebraically slice, genus one knot has (at most) three first-order L^2 -signatures. We will discuss the relationship between our higher-order genus invariant and these first-order L^2 -signatures.

2.1 Motivation and definition

Let Σ_g be a compact, oriented surface with one boundary component. If $f : \Sigma_g \hookrightarrow S^3$ is an embedding with $K = f(\partial\Sigma_g)$, then some invariants of K can be computed using this embedded surface $f(\Sigma_g)$. Such a surface is called a Seifert surface for K . For example, any Seifert surface can be used to compute the knot's Alexander polynomial. This polynomial is encoded in the knot Floer homology $\widehat{HFK}(K)$. The smallest genus of such embedded surfaces with boundary K is called the genus of K , $g(K)$, and this invariant is also detected by $\widehat{HFK}(K)$. We naïvely ask whether $\widehat{HFK}(K)$ contains anymore information about the embedded surfaces with boundary K . For example, we are interested in whether $\widehat{HFK}(K)$ contains information about the knottedness of certain simple closed curves on Seifert surfaces $f(\Sigma_g)$ for K . We

will restrict our attention to genus one, algebraically slice knots.

Our motivating example is the positively-clasped, untwisted Whitehead double of a knot K , denoted $D(K)$ and depicted in Figure 2.1. There is an obvious genus one Seifert surface for $D(K)$. In [Hor2], the author defined a knot invariant, g_1 (not to be confused with the 1st-order genus defined in Chapter 4 of this thesis), that measures the knottedness of the bands in Seifert surface for a knot. For many knots K , this invariant applied to $D(K)$ “detects” the genus of K , i.e. $g_1(D(K)) = 1 + g(K)$. One may ask if $\widehat{HFK}(D(K))$ detects $g_1(D(K)) \approx g(K)$, and by Hedden’s formula [Hed1, Theorem 1.2], the answer is “yes” in the sense that $\widehat{HFK}(D(K), 1)$ has as a direct summand $\bigoplus_{j=-g(K)}^{g(K)} G_j(K)$, where the $G_j(K)$ are certain groups depending on K . Due to computational difficulties, it is unknown whether $\widehat{HFK}(K)$ detects $g_1(K)$ in general. We aim to define an invariant that is more computable than g_1 and which measures, more or less, the same thing.

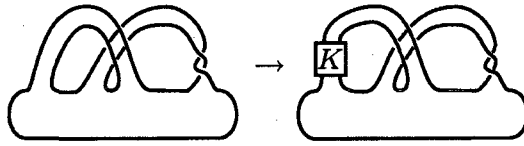


Figure 2.1: $D(K)$: the positively-clasped, untwisted Whitehead double of K

Definition 2.2. Let K be an algebraically slice knot in S^3 of genus one. Let Σ be any genus one Seifert surface for K . Then Σ has a **metabolizer** \mathfrak{m} , a rank one submodule of $H_1(\Sigma; \mathbb{Z})$ on which the Seifert form vanishes. One can show that Σ has exactly two metabolizers \mathfrak{m}_1 and \mathfrak{m}_2 . Let $[\alpha_1]$ and $[\alpha_2] \in H_1(\Sigma; \mathbb{Z})$ be generators of \mathfrak{m}_1

and \mathfrak{m}_2 , respectively. By the classification of essential closed curves on a punctured torus [Min], each $[\alpha_i]$ determines a unique oriented knot in Σ ; denote this knot by α_i . The knot α_i is called a **derivative of K with respect to the metabolizer \mathfrak{m}_i** .

To sum up, each genus one Seifert surface Σ for an algebraically slice knot K has exactly two (up to orientation) derivatives α_1 and α_2 . We denote this set of derivatives as $\partial(K, \Sigma) = \{\alpha_1, \alpha_2\}$.

Let $\mathcal{G}(K)$ denote the set of isotopy classes (in $S^3 - K$) of genus one Seifert surfaces for K , and if α is a null-homologous knot in $S^3 - K$, let $g^K(\alpha)$ denote the genus of α in $S^3 - K$. We define the **differential genus** of K to be

$$\text{dg}(K) = \min_{\Sigma \in \mathcal{G}(K)} \left\{ \max_{\partial(K, \Sigma) = \{\alpha_1, \alpha_2\}} \{g^K(\alpha_1), g^K(\alpha_2)\} \right\}$$

Remark 2.3. The differential genus measures the knottedness of self-linking zero curves on (genus one) Seifert surfaces for K . One may define metabolizers and derivatives of algebraically slice knots of higher genus (see [CHL2]), but in the higher genus setting, a metabolizer may have infinitely many distinct derivatives. One may try to generalize the definition of differential genus to higher genus algebraically slice knots; this may be taken up in a future paper.

2.2 Examples

Example 2.4. Let K be a knot that is non-trivial and not a cable. By [Whi], the untwisted Whitehead double of K , $D(K)$, has a unique minimal genus Seifert surface. Each of the untwisted curves on this Seifert surface have the same knot type as K . One can further argue that $\text{dg}(D(K)) = g(K)$.

Example 2.5. Let $R = 9_{46}$ as depicted in Figure 2.2. A symplectic basis of curves α and β have been drawn for the implied Seifert surface Σ . One can check that α and β have self-linking zero, and so the two derivatives for Σ are α and β . Each of α and β is unknotted, however $g^R(\alpha) = g^R(\beta) = 1$. The knot Floer homology of R is

$$\begin{array}{ccc} 0 & 0 & 2 \end{array}$$

$$\begin{array}{ccc} 0 & 5 & 0 \end{array}$$

$$\begin{array}{ccc} 2 & 0 & 0 \end{array}$$

where the 5 appears in bigrading $(0, 0)$. The genus of R is one, and since $\text{rank } \widehat{HFK}(R, 1) = 2 < 4$, we may apply Theorem 2.3 of [Juh] to conclude that Σ is the unique genus one Seifert surface for R up to isotopy. We conclude that $\text{dg}(R) = 1$.

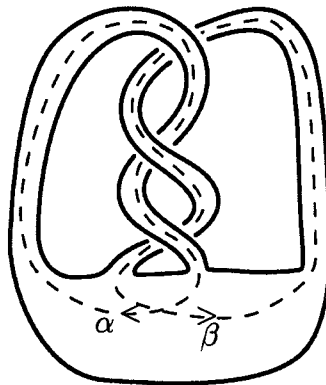


Figure 2.2: The 9_{46} knot

Example 2.6. Now consider the knot K_n in Figure 2.3, where $n \in \mathbb{N}$. Observe that $K_0 = 9_{46}$. A symplectic basis of curves x and y have been drawn for the implied

Seifert surface F . One can check the Seifert form of F to be

$$\begin{pmatrix} 3n & -2 \\ -1 & 0 \end{pmatrix}$$

The two curves of self-linking zero are $\alpha_n = x + ny$ and $\beta_n = y$. As in the calculation for 9_{46} , one can check that $g^{K_n}(\beta_n) = 1$. The other curve α_n is more complicated; see Figure 2.6. The knot α_n can be represented by the braid on $n + 1$ strands depicted in Figure 2.6.

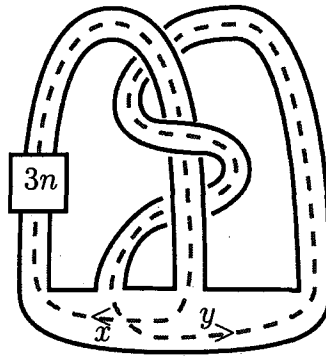


Figure 2.3: A diagram for K_n , and a basis for a Seifert surface

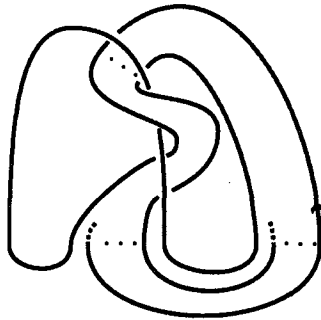


Figure 2.4: A knot diagram of α_n , a curve of self-linking zero

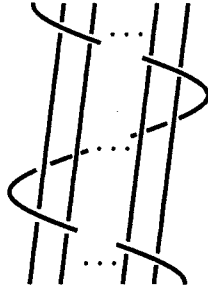


Figure 2.5: A braid representative of α_n using $n + 1$ strands

By [Cro, Corollary 4.1], the Seifert surface constructed by applying Seifert's algorithm to the braid diagram in Figure 2.6 has minimal genus. In particular $g(\alpha_n) = n$, and hence $g^{K_n}(\alpha_n) \geq n$. We must argue that $\text{dg}(K_n) \geq n$. For a given n , one may easily construct a grid diagram for K_n . For several small values of n , we used Marc Culler's Gridlink [Cul] to compute the knot Floer homology of K_n . We found that $\widehat{HFK}(K_n) \cong \widehat{HFK}(9_{46})$ for these values of n (although this family is defined for $n \in \mathbb{N}$, we verified the computation for $n = -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1$). A recent result of M. Hedden [Hed2] implies that $\widehat{HFK}(K_n) \cong \widehat{HFK}(9_{46})$ for all n . By [Juh], F is the unique genus one Seifert surface for K_n . By previous arguments, we conclude that $\text{dg}(K_n) = g^{K_n}(\alpha_n) \geq g(\alpha_n) \geq n$.

Theorem 2.1 follows.

Remark 2.7. One can show by calculating the Alexander polynomials of the unknotted curves for $K_{1/3}$ that $\text{dg}(K_{1/3}) \geq 2$. Thus, $K_{1/3}$ is an explicit example of a knot with the same knot Floer homology as 9_{46} and distinct differential genus.

Theorem 2.8. *There exists an infinite family of knots K_n such that*

- $\widehat{HFK}(K_n) \cong \widehat{HFK}(K_m)$ for all m and n , and
- $\text{dg}(K_n) \neq \text{dg}(K_m)$ for $m \neq n$.

Proof. The family is constructed by taking a subsequence of the knots K_n from Example 2.6. □

2.3 First-order L^2 -signatures and the differential genus

Metabelian signatures of knots have been defined by Casson-Gordon, Letsche, Cochran-Orr-Teichner, Friedl, and Cochran-Harvey-Leidy [CG1][CG2][Let][COT1][Fri][CHL2]. We are interested in those of Cochran, Harvey, and Leidy because each genus one, algebraically slice knot has two “first-order L^2 -signatures.” We now recall some of the background needed to define these signatures.

Suppose K is an oriented knot in S^3 , M_K denote the closed, oriented 3-manifold obtained by zero-surgery on K , and $G = \pi_1(M_K)$. Let $G^{(1)}$ denote the commutator subgroup of G and $G^{(2)}$ the commutator subgroup of $G^{(1)}$. The **classical rational Alexander module** of K is

$$\mathcal{A}_0(K) := \frac{G^{(1)}}{G^{(2)}} \bigotimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Q}[t, t^{-1}]$$

Here $G^{(1)}/G^{(2)}$ is identified with the classical Alexander module $H_1(M_K; \mathbb{Z}[t, t^{-1}])$.

The **Blanchfield pairing** of K

$$\mathcal{B}\ell_0^K : \mathcal{A}_0(K) \times \mathcal{A}_0(K) \rightarrow \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]$$

is defined by

$$\mathcal{B}\ell_0^K(x, y) = \sum_{n \in \mathbb{Z}} \frac{(d \cdot yt^n)t^n}{\Delta_K(t)}$$

where $\Delta_K(t)$ is the Alexander polynomial of K and d is a 2-chain with $\partial d = \Delta_K(t) \cdot x$.

We say a submodule $P \subset \mathcal{A}_0(K)$ is **Lagrangian** (respectively **isotropic**) if $P = P^\perp$ (respectively $P \subset P^\perp$) with respect to the Blanchfield pairing. To a submodule $P \subset \mathcal{A}_0(K)$, we can associate a metabelian quotient $\phi_P : G \rightarrow G/\tilde{P}$ by setting $\tilde{P} = \ker (G^{(1)} \rightarrow G^{(1)}/G^{(2)} \rightarrow \mathcal{A}_0(K) \rightarrow \mathcal{A}_0(K)/P)$. To this quotient we can associate a real number, called the Cheeger-Gromov von Neumann ρ -invariant, $\rho(M_K, \phi_P)$ [CG3] (see Chapter 3 for a description).

Definition 2.9. The **first-order L^2 -signatures of a knot K** are the real numbers $\rho(M_K, \phi_P)$ where P is a Lagrangian submodule of $\mathcal{A}_0(K)$ with respect to $\mathcal{B}\ell_0^K$.

Remark 2.10. These are a subset of the metabelian L^2 -signatures of Cochran, Harvey, and Leidy [CHL2, Definition 4.1], who allow for P to be isotropic.

Assume K is a genus one, algebraically slice knot with a Seifert surface Σ . The reader will recall that $H_1(\Sigma; \mathbb{Z})$ generates $\mathcal{A}_0(K)$ as a $\mathbb{Q}[t, t^{-1}]$ -module (one must pick a lift of Σ to the infinite cyclic cover). If $\Delta_K(t) = 1$, then $\mathcal{A}_0(K) = 0$ has no Lagrangian submodules. On the other hand, if $\Delta_K(t) \neq 1$, then $\Delta_K(t) = f(t)f(t^{-1})$ for some linear polynomial $f(t)$. $\mathcal{A}_0(K)$ must be isomorphic to $\frac{\mathbb{Q}[t, t^{-1}]}{\langle f(t)f(t^{-1}) \rangle}$. Thus, any proper submodule P must be

$$\frac{\mathbb{Q}[t, t^{-1}]}{\langle f(t) \rangle} \quad \text{or} \quad \frac{\mathbb{Q}[t, t^{-1}]}{\langle f(t^{-1}) \rangle}$$

Since the Blanchfield pairing is primitive, $\mathcal{A}_0(K)$ will have precisely two Lagrangians.

By the Definitions 2.2, K will have precisely two Lagrangians and hence two first-order L^2 -signatures.

Definition 2.11. Suppose $P \subset \mathcal{A}_0(K)$ is a Lagrangian. The metabolizer \mathfrak{m} represents P if the image of \mathfrak{m} under the map

$$i_* \circ (\text{id} \otimes 1) : H_1(\Sigma; \mathbb{Z}) \hookrightarrow H_1(\Sigma; \mathbb{Z}) \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}_0(K)$$

spans P as a \mathbb{Q} -vector space. (To define i_* , it is necessary to choose a lift of Σ to the infinite cyclic cover, but this definition is independent of the choice).

Proposition 2.12 (Lemma 5.5 of [CHL2]). *Let K be an algebraically slice knot and P be a Lagrangian of $\mathcal{A}_0(K)$. If Σ is any Seifert surface for K , then some metabolizer of $H_1(\Sigma)$ represents P .*

Proposition 2.13 (Corollary 5.8 of [CHL2]). *Let K be a genus one, algebraically slice knot. Suppose P is a Lagrangian for K , Σ is a genus one Seifert surface for K , \mathfrak{m} is a metabolizer of Σ representing P , and J is a derivative with respect to \mathfrak{m} . Then the first-order L^2 -signature of K with respect to P is equal to $\rho_0(J) = \int_{S^1} \sigma_\omega(J) d\omega$, the integral of the Levine-Tristram signature function.*

Determining $\text{dg}(K)$ involves computing the genus of two curves from each genus one Seifert surface, of which there may be many. Examples of knots that have an arbitrary number of non-isotopic Seifert surfaces are known [Suz, p. 47]. Yet we have the following remarkable fact: if just one of the first-order L^2 -signatures is large, then the differential genus must be large.

Proposition 2.14. *Let K be a genus one, algebraically slice knot with non-trivial Alexander polynomial. Let ρ_1 and ρ_2 denote the first-order L^2 -signatures of K with respect to the two Lagrangians P_1 and P_2 . Then $2 \operatorname{dg}(K) \geq \max\{|\rho_1|, |\rho_2|\}$.*

Proof. Let Σ be the Seifert surface where the minimum is attained. For either derivative $J_i \subset \Sigma$, where J_i represents the Lagrangian P_i , we have

$$2 \operatorname{dg}(K) \geq 2 g^K(J_i) \geq 2 g(J_i) \geq \left| \int_{S^1} \sigma_\omega(J_i) d\omega \right| \geq |\rho_i|$$

□

Chapter 3

Higher-order signatures of knots

Given a 4-manifold X and a homomorphism $\varphi : \pi_1(X) \rightarrow \Gamma$, where Γ is a PTFA group, Cochran, Orr, and Teichner defined the L^2 -signature of the pair (X, φ) (see Definition 1.3). The goal of this chapter is to assign to an (n) -solvable knot K a set of higher-order signatures $\mathfrak{S}^n(K)$. We will show in Chapter 4 that one of these higher-order signatures bounds the higher-order genus from below.

If M is a closed, oriented 3-manifold, Γ a discrete group, and $\phi : \pi_1(M) \rightarrow \Gamma$ a homomorphism, J. Cheeger and M. Gromov [CG3] defined the **von Neumann ρ -invariant**, $\rho(M, \phi) \in \mathbb{R}$. They first picked a Riemannian metric g on M and defined η -invariants of (M, g) and the covering space determined by ϕ , and they proved the difference of the η -invariants is independent of the metric. Cochran and Teichner [CT] give a brief, analytical overview of the von Neumann ρ -invariants.

A critical tool in Chapter 4 will be the Cheeger-Gromov estimate: given a closed, oriented 3-manifold M , there is a constant $C_M > 0$ such that

$$|\rho(M, \phi)| < C_M \quad (3.1)$$

for any discrete group Γ and any homomorphism $\phi : \pi_1(M) \rightarrow \Gamma$.

3.1 Definition

Definition 3.1. For $K \in \mathcal{F}_n$, we define the n^{th} -order signatures of K to be the elements of the set $\mathfrak{S}^n(K) = \left\{ \rho(M_K, \phi) \in \mathbb{R} \mid \phi : \pi(M_K) \xrightarrow{i_*} \pi \twoheadrightarrow \pi/\pi_r^{(n+1)} \right\}$ where $\pi = \pi_1(W)$, W is an (n) -solution for M_K , $i : M_K \rightarrow W$ is the inclusion map, and $\rho(M_K, \phi)$ is the associated von Neumann ρ -invariant.

The set of n^{th} -order signatures of K is an isotopy invariant, since the von Neumann ρ -invariant depends on the homeomorphism type of M_K and the homomorphism ϕ .

The relationship between von Neumann ρ -invariants and L^2 -signatures is surprisingly simple in the present context. If $(M, \phi) = \partial(W, \varphi)$ for some compact, oriented 4-manifold W and homomorphism $\varphi : \pi_1(W) \rightarrow \Gamma$, then

$$\rho(M, \phi) = \sigma^{(2)}(W, \varphi) - \sigma(W) \quad (3.2)$$

where $\sigma(W)$ is the ordinary signature of W . If W is an (n) -solution for $M = M_K$ and ϕ is as in Definition 3.1, then $(M, \phi) = \partial(W, \varphi)$, and so 3.2 holds. By the definition of an (n) -solution, the ordinary signature of W vanishes, and thus if $\rho(M, \phi)$ is the n^{th} -order signature of K corresponding to the (n) -solution W , we have

$$\rho(M_K, \phi) = \sigma^{(2)}(W, \varphi) \quad (3.3)$$

In other words, the set of n^{th} -order signatures of a knot K is equal to the set of L^2 -signatures associated to the (n) -solutions of M_K and their quotients modulo the $(n+1)^{\text{th}}$ term of the rational derived series. Since we have expressed the n^{th} -order signatures as signatures of 4-manifolds, one may be tempted to believe that $\mathfrak{S}^n(K)$ is a concordance invariant. We disprove this with a counterexample (Example 3.2).

3.2 Infection

The construction of many examples in this thesis relies on a technique known as **infection**. Let R be a fixed knot or link and T be a fixed knot in S^3 . Suppose α is a simple closed curve in $S^3 - R$ such that α is itself the unknot. Some number m of strands of R pierce the disc bounded by α . Let $R(\alpha, T)$ denote the knot obtained by replacing the m trivial strands of R by m strands “tied into the knot T .” See Figure 3.1. More precisely, one obtains $R(\alpha, T)$ by removing a regular neighborhood of α and gluing in $S^3 - T$ in such a way that identifies the meridian of α with the longitude of T and the longitude of α with the meridian of T . We say $R(\alpha, T)$ is the result of infecting R by T along α .

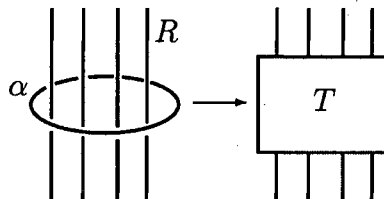


Figure 3.1: Infecting R by T along α

If R is a ribbon knot, $\alpha \subset \pi_1(S^3 - N(R))^{(p)}$, and $T \in \mathcal{F}_q$, then $R(\alpha, T) \in \mathcal{F}_{p+q}$ [CHL3, Lemma 6.4].

Example 3.2. Let U denote the unknot. Then $M_U \cong S^1 \times S^2$, and $\pi_1(M_U) \cong \mathbb{Z}$. If W is any (n) -solution for M_U and $\pi = \pi_1(W)$, then the homomorphism $\phi : \pi_1(M) \rightarrow \pi \twoheadrightarrow \pi/\pi_r^{(n+1)}$ will factor through $H_1(W) \cong \mathbb{Z}$. By [COT1][Proposition 5.13], $\sigma^{(2)}(W, \pi/\pi_r^{(n+1)}) = \sigma^{(2)}(W, \mathbb{Z})$, and we conclude

$$\rho(M_U, \phi) = \sigma^{(2)}(W, \pi/\pi_r^{(n+1)}) = \sigma^{(2)}(W, \mathbb{Z}) = \rho_0(U) = \int_{S^1} \sigma_\omega(U) d\omega = 0$$

by [COT1]. We conclude $\mathfrak{S}^n(U) = \{0\}$.

Let $R = 9_{46}$ with curves α and $\beta \subset S^3 - N(R)$ (Figure 4.3), and denote the left-handed trefoil by T . Let K denote the knot obtained by infecting R along α by $T\#T$ and along β by any ribbon knot S . We will construct a non-zero element of $\mathfrak{S}^1(K)$ in the case S is the unknot, where no infection is done along β . Since R is ribbon, $\alpha \in \pi_1(S^3 - N(R))^{(1)}$, and $T\#T \in \mathcal{F}_0$, our knot K is (1) -solvable. In fact, K is a slice knot (cut the $T\#T$ band). We will construct an explicit (1) -solution, V , and use it to produce a non-zero 1st-order signature of K .

Let X denote the slice disc complement for R in which β bounds the disc cutting the right-hand band of R , and let W be any (0) -solution for $T\#T$. Form a 4-manifold

$$E = M_R \times [0, 1] \bigcup_{N(\alpha) \times \{1\} = N(T\#T) \times \{0\}} -M_{T\#T} \times [0, 1]$$

Then $\partial E = M_R \sqcup M_{T\#T} \sqcup -M_K$. Then $V = -(E \cup X \cup W)$ is a (1) -solution for M_K by [CT, Proof of Theorem 4.2]. Let $\pi = \pi_1(V)$. Furthermore, the homomorphism $\phi : \pi_1(M_K) \rightarrow \pi/\pi_r^{(2)}$ induces a homomorphism $\phi_R : \pi_1(M_R) \rightarrow \pi/\pi_r^{(2)}$, and according

to [CT, Proposition 4.4], the von Neumann ρ -invariants are related by

$$\rho(M_R, \phi_R) - \rho(M_K, \phi) = \epsilon \rho_0(T \# T)$$

where $\epsilon = 0$ or 1 according to whether $\phi_R([\alpha]) = 1$ or not.

Recall that β bounds a disc in X , and so $[\beta] = 1 \in \pi_1(V)$. By the argument in Example 4.14, α does not map into $\pi_r^{(2)}$. This implies that $\phi_R([\alpha]) \neq 1$, and so $\rho(M_R, \phi_R) - \rho(M_K, \phi) = \frac{8}{3}$. Thus, $\rho(M_K, \phi) \neq 0$ as long as $\rho(M_R, \phi_R) \neq \frac{8}{3}$. If this ρ -invariant of M_R is $8/3$, we could repeat the above construction with an some even number of connected sums of T . Since the set of von Neumann ρ -invariants of M_R is bounded (cf. equation 3.1), we can infect with enough copies of $T \# T$ to produce some slice knot K with some non-zero 1st-order signature.

Since K is concordant to U and $\mathfrak{S}^1(K) \neq \mathfrak{S}^1(U)$, the set of 1st-order signatures is not a concordance invariant.

3.3 Construction of knots with large higher-order signatures

In Example 3.2, we produced a knot in \mathcal{F}_1 , one of whose 1st-order signatures was non-zero. We could have just as easily produced a knot in \mathcal{F}_1 , one of whose 1st-order signatures is arbitrarily large. In the following theorem, we prove a more general statement, namely that there is a knot in \mathcal{F}_n , *all of whose* n^{th} -order signatures are arbitrarily large.

Theorem 3.3. *Let $C > 0$. There is a knot $K \in \mathcal{F}_n$ such that every n^{th} -order signature $\rho \in \mathfrak{S}^n(K)$ satisfies $|\rho| > C$.*

The proof relies heavily on a theorem of Cochran and Teichner.

Theorem 3.4 (4.3 of [CT]). *Suppose R is a genus two fibered ribbon knot. Then for any $n \geq 1$, there is an oriented trivial link $\{\eta_1, \dots, \eta_k\}$ in $S^3 - N(R)$ such that*

$$[\eta_i] \in \pi_1(M_R)^{(n)}, \text{ for } 1 \leq i \leq k,$$

and for any (n) -solution V of M_R , there is some i such that $j_([\eta_i]) \notin \pi_1(V)_r^{(n+1)}$, where $j : M_R \rightarrow V$ is the inclusion map.*

Proof of Theorem 3.3. Let $C > 0$, and let R be a genus two fibered ribbon knot. Let η_i , $1 \leq i \leq k$, as in Theorem 3.4. Let J denote the (0) -solvable knot in Figure 3.2 from [CT], where it was proven that $\rho_0(J) = \frac{4}{3}$.

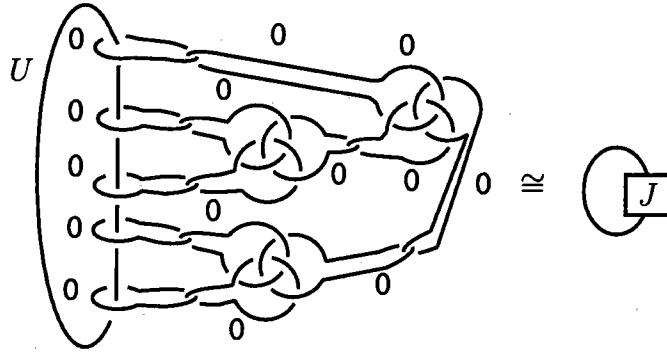


Figure 3.2: The knot J as surgery on an unknot U

Let C_{M_R} denote the Cheeger-Gromov constant of M_R (cf. equation 3.1), and pick

$m \geq 1$ such that

$$\frac{4m}{3} > C + C_{M_R}$$

Denote $J_m = \#_{i=1}^m J$, and observe that $\rho_0(J_m) = \frac{4m}{3}$. Let K denote the result obtained by infecting R along each η_i using the knot J_m . Then K is (n) -solvable.

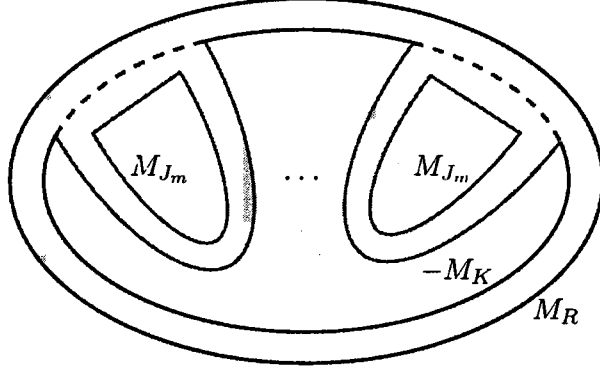
Since our J_m are (0) -solvable, let W_m denote a (0) -solution for J_m . We form a 4-manifold E from

$$M_R \times [0, 1] \bigsqcup_{i=1}^j -M_{J_m} \times [0, 1]$$

by identifying, for each i , the copy of $\eta_i \times D^2$ in $M_R \times \{1\}$ with the tubular neighborhood of J_m in $M_{J_m} \times \{0\}$ as in Figure 3.3. The dashed arcs represent the solid tori $\eta_i \times D^2$. As indicated in Figure 3.3, $\partial E = M_R \sqcup -M_K \sqcup M_{J_m} \sqcup \cdots \sqcup M_{J_m}$. We form another 4-manifold C from E by gluing a copy of W_m to each $M_{J_m} \subset \partial E$.

Now let W be any (n) -solution for M_K . Let $V = C \cup_{-M_K} -W$ so that $\partial V = M_R$. Then V is an (n) -solution for M_R [CT, Proof of Theorem 4.2]. From our previous discussion, there is a η_k with $i_*([\eta_k]) \notin \pi_1(V)_r^{(n+1)}$. Since η_k lives in M_K , we may include η_k into W . Since $W \subset V$, $i_*([\eta_k]) \notin \pi_1(W)_r^{(n+1)}$.

Consider the homomorphism $\phi : \pi_1(M_K) \xrightarrow{i_*} \pi_1(W) \xrightarrow{\pi} \pi_1(W)/\pi_1(W)_r^{(n+1)}$. Let $\Gamma = \pi_1(W)/\pi_1(W)_r^{(n+1)}$. Now $M_R - (\sqcup \eta_i) \subset M_K$, so ϕ induces a homomorphism $\phi' : \pi_1(M_R - (\sqcup \eta_i)) \rightarrow \Gamma$. Since M_R is obtained by $M_R - (\sqcup \eta_i)$ by adding j 2-cells along the meridians of the η_i and then by adding j 3-cells, this ϕ' will extend to a homomorphism $\phi_R : \pi_1(M_R) \rightarrow \Gamma$ if the meridians of the η_i die under ϕ . Now $\eta_i \in \pi_1(M_R)^{(n)}$ and $\Gamma^{(n+1)} = 1$, so [Coc, Theorem 8.1] implies that $\eta_i \in \pi_1(M_R)^{(n)}$. Since the meridian μ_i of each J_m is identified with the longitude of η_i ,

Figure 3.3: The 4-manifold E

$\mu_i \in \pi_1(M_K)^{(n)}$. Thus $\phi(\mu_i) \in \Gamma^{(n)}$. Since μ_i generates $\pi_1(S^3 - J_m)/\pi_1(S^3 - J_m)^{(1)}$, we see $\phi\left(\pi_1(S^3 - J_m)^{(1)}\right) \subset \Gamma^{(n+1)} = 1$. In particular the meridian of each η_i dies under ϕ , and hence ϕ' extends to a map $\phi_R : \pi_1(M_R) \rightarrow \Gamma$.

By [CT, Proposition 4.4], the ρ -invariants of M_K and M_R are related by

$$\rho(M_R, \phi_R) - \rho(M_K, \phi) = \sum_{i=1}^j \epsilon_i \rho_0(J_m)$$

where $\epsilon_i = 0$ or 1 according to whether $\phi_R([\eta_i]) = 1$ or not. We argued that previously that $i_*([\eta_k]) \notin \pi_1(W)_r^{(n+1)}$, so $\phi_R([\eta_i]) \neq 1$. Thus, for some positive integer c , we have

$$|\rho(M_K, \phi)| = \left| \frac{4mc}{3} - \rho(M_R, \phi_R) \right|$$

Observe that $\frac{4m}{3} > C + C_{M_R} > C + |\rho(M_R, \phi_R)|$, and so $\frac{4m}{3} - \rho(M_R, \phi_R) > C$. We conclude that $|\rho(M_K, \phi)| > C$.

Since W was an arbitrary (n) -solution for K , we have proved that every n^{th} -order signature for K is larger than C in absolute value. \square

We provide another method of building knots with large higher-order signatures.

This method takes the form of an additivity result for the higher-order signatures.

Theorem 3.5. *If $\rho_1 \in \mathfrak{S}^n(K_1)$ and $\rho_2 \in \mathfrak{S}^n(K_2)$, then $\rho_1 + \rho_2 \in \mathfrak{S}^n(K_1 \# K_2)$.*

Proof. Let M_i denote 0-surgery on K_i , and let M denote 0-surgery on $K_1 \# K_2$. Let W_i be an (n) -solution for M_i , and define $\phi_i : \pi_1(M_i) \rightarrow \mathbf{1}(W_i) \rightarrow \pi_1(W_i)/\pi_1(W_i)_r^{(n+1)}$. Let $\rho_i = \rho(M_i, \phi_i)$.

We construct a 4-manifold V with $\partial V = M_1 \sqcup M_2 \sqcup -M$ as follows. Attach to $M_1 \sqcup M_2 \times [0, 1]$ a 1-handle. The boundary of the result is $M_1 \sqcup M_2 \sqcup -(M_1 \# M_2)$. To $-(M_1 \# M_2)$, attach a 0-framed 2-handle as depicted in Figure 3.4. The result after adding the 1-handle and 2-handle will be called V .

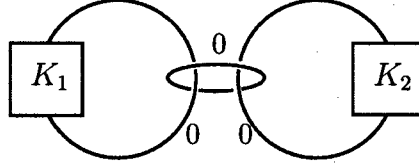


Figure 3.4: Adding a 2-handle

Let $W = V \cup_{M_1} W_1 \cup_{M_2} W_2$. It is known that W is an (n) -solution for $M = M_{K_1 \# K_2}$.

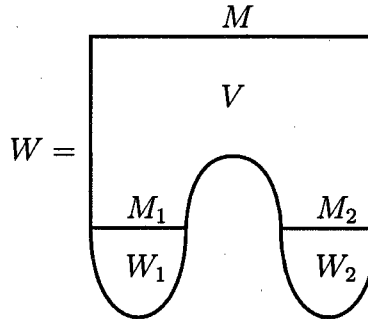


Figure 3.5: A schematic diagram of the 4-manifolds V and W

Let $A = \pi_1(W_1)$, $B = \pi_1(W_2)$, and $G = \pi_1(W) \cong \pi_1(V)$. We may pick a generator μ_1 of A that maps to a generator of $H_1(W_1)$ and a generator μ_2 of B that maps to a generator of $H_1(W_2)$, so that $G \cong A * B / \langle \mu_1 = \mu_2 \rangle \cong A *_Z B$. Let

$\psi : \pi_1(V) \rightarrow G/G_r^{(n+1)}$ be the composition of the inclusion-induced map

$\pi_1(V) \rightarrow \pi_1(W) = G$ and the projection $G \twoheadrightarrow G/G_r^{(n+1)}$. We have

$$\sigma^{(2)}(V, \psi) - \sigma(V) = \rho(M_1, \psi_1) + \rho(M_2, \psi_2) - \rho(M, \psi_3) \quad (3.4)$$

where $\psi_i : \pi_1(M_i) \rightarrow \pi_1(V) \rightarrow \pi_1(W) = G \twoheadrightarrow G/G_r^{(n+1)}$ for $i = 1, 2$ and

$\psi_3 : \pi_1(M) \xrightarrow{\cong} \pi_1(V) \rightarrow \pi_1(W) = G \twoheadrightarrow G/G_r^{(n+1)}$.

We claim the following:

- (a) $\sigma^{(2)}(V, \psi) - \sigma(V) = 0$,
- (b) $\rho(M_i, \psi_i) = \rho(M_i, \phi_i)$ for $i = 1, 2$, and
- (c) $\rho(M, \psi_3) \in \mathfrak{S}^n(K_1 \# K_2)$.

After proving (a), (b), and (c), the proof is complete.

To prove (a), let $M_- = M_1 \sqcup M_2$ and $\Gamma_n = G/G_r^{(n+1)}$. Since Γ_n is PTFA, $\mathbb{Z}\Gamma_n$ is an Ore domain and hence admits a classical (right) ring of quotients $\mathcal{K}\Gamma_n$ into which $\mathbb{Z}\Gamma_n$ embeds [Pas, pp 591–592]. Hence, a finitely generated (right) module M over $\mathbb{Z}\Gamma_n$ has a well-defined rank, namely, the rank of the vector space $M \otimes_{\mathbb{Z}\Gamma_n} \mathcal{K}\Gamma_n$ [Coh, p 48]. From the long exact sequence for the pair (V, M_-) with coefficients in $\mathcal{K}\Gamma_n$, we obtain the following short exact sequence:

$$0 \rightarrow \text{im}(\iota) \rightarrow H_2(V, M_-) \rightarrow H_1(M_-) \rightarrow H_1(V) \rightarrow H_1(V, M_-) \rightarrow 0$$

where $\iota : H_2(V) \rightarrow H_2(V, M_-)$ is induced by inclusion. Since V is obtained from $M_- \times I$ by attaching one 1-handle and one 2-handle, there is a cellular chain complex for the pair (V, M_-)

$$0 \rightarrow \mathcal{K}\Gamma_n \rightarrow \mathcal{K}\Gamma_n \rightarrow 0$$

where the $\mathcal{K}\Gamma_n$'s lie in homological gradings 1 and 2. Thus $\text{rank}_{\mathcal{K}\Gamma_n} H_2(V, M_-) - \text{rank}_{\mathcal{K}\Gamma_n} H_1(V, M_-) = 0$. By [COT1, Proposition 2.11], $\text{rank}_{\mathcal{K}\Gamma_n} H_1(M_-) = \text{rank}_{\mathcal{K}\Gamma_n} H_1(V) = 0$. Thus, $\text{rank}_{\mathcal{K}\Gamma_n} \text{im}(\iota) = 0$, and since $\mathcal{K}\Gamma_n$ is a (non-commutative) field, ι must be the zero homomorphism. By the long exact sequence for the pair (V, M_-) , the inclusion-induced map $H_2(M_-) \rightarrow H_2(V)$ is surjective. Thus, $\sigma^{(2)}(V, \psi) = \sigma(V) = 0$, and the proof of (a) is complete.

To prove (b), we first note that $\psi_1 = i_A \circ \phi_1$ where $i_A : A/A_r^{(n+1)} \rightarrow G/G_r^{(n+1)}$ is induced by the inclusion $W_1 \rightarrow W$. In fact, we claim i_A is a monomorphism, and it follows from [Har, Lemma 3.7] that $\rho(M_1, \psi_1) = \rho(M_1, i_A \circ \phi_1) = \rho(M_1, \phi_1)$. Let $\omega : B \rightarrow A$ be the composition of the maps $B \twoheadrightarrow \mathbb{Z} = \langle \mu_2 \rangle \xrightarrow{\cong} \langle \mu_2 \rangle \hookrightarrow A$. By the universal property of the free product of groups, there is a unique homomorphism $\xi : A * B \rightarrow A$ such that $\xi \circ i = \omega$. Let π denote projection of $A * B \twoheadrightarrow G$. Since $\ker(\pi) = \langle \mu_1 \mu_2^{-1} \rangle < \ker(\xi)$, there is a unique homomorphism $r_A : G \rightarrow A$ such that $r_A \circ \pi = \xi$.

$$\begin{array}{ccccc}
 & & \pi & & \\
 & & \longleftarrow & & \\
 G & \xleftarrow{\pi} & A * B & \xleftarrow{i} & B \\
 & \searrow \scriptstyle r_A & \downarrow \scriptstyle \xi & \swarrow \scriptstyle \omega & \\
 & & A & &
 \end{array}$$

If i_A denotes the inclusion map of $A \hookrightarrow G$, one can check that $r_A \circ i_A = \text{id}_A$, i.e. $r_A : G \rightarrow A$ is a retract. Since group homomorphisms map the n^{th} term of the

rational derived series of the domain into the n^{th} term of the rational derived series of the codomain, we have induced maps $i_A : A/A_r^{(n+1)} \rightarrow G/G_r^{(n+1)}$ and $r_A : G/G_r^{(n+1)} \rightarrow A/A_r^{(n+1)}$ such that $r_A \circ i_A = \text{id}_{A/A_r^{(n+1)}}$. Thus $i_A : A/A_r^{(n+1)} \rightarrow G/G_r^{(n+1)}$ is a monomorphism, as desired.

Claim (c) follows from the fact that $\psi_3 : \pi_1(M) \rightarrow G/G_r^{(n+1)}$ is the composition of the inclusion-induced map (from M into the (n) -solution W) and projection onto $G/G_r^{(n+1)}$.

□

Chapter 4

Higher-order genera of knots

Cochran, Orr, and Teichner have introduced two filtrations of the topological knot concordance group \mathcal{C} [COT1]. The (n) -solvable filtration

$$\cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

is defined in terms of algebraic properties on the second homology of certain 4-manifolds, each of whose boundary is 0-surgery on a knot. The Grope filtration

$$\cdots \subset \mathcal{G}_{n+2.5} \subset \mathcal{G}_{n+2} \subset \cdots \subset \mathcal{G}_3 \subset \mathcal{G}_{2.5} \subset \mathcal{G}_2 \subset \mathcal{C}$$

is defined much more geometrically. Rigorous definitions of these filtrations were provided in the introduction. These filtrations are related to one another in the sense that $\mathcal{G}_{n+2} \subset \mathcal{F}_n$ for all $n \in \frac{1}{2}\mathbb{N}$ [COT1, Theorem 8.11]. Recently, Cochran, Harvey, and Leidy proved that $\mathcal{F}_n/\mathcal{F}_{n.5}$ has infinite rank for all n [CHL4]. Subsequently, the author proved the analogous result for the Grope filtration [Hor1]. These results were proven using signatures of certain 4-manifolds. While algebraic techniques are appropriate when working with the (n) -solvable filtration, they do not reflect the

geometric nature of the Grope filtration. The main focus of this chapter is to define a geometric invariant that will distinguish knots in \mathcal{G}_{n+2} .

In the spirit of the Cochran-Orr-Teichner filtrations of \mathcal{C} , we introduce a series of refinements of the slice genus. For knots in \mathcal{G}_{n+2} , we will define a concordance invariant called the n^{th} -order genus.

Definition 4.1. For $K \in \mathcal{G}_{n+2}$, define the n^{th} -order genus of K to be the minimum of the genera of the first-stage surfaces of Grope of height $n + 2$ in D^4 bounded by K . Denote the n^{th} -order genus of K by $g_n(K)$. With this numbering scheme, the slice genus of K is the $(-1)^{\text{st}}$ -order genus of K .

It is immediately clear that for $K \in \mathcal{G}_{n+2}$, $0 \leq g_{-1}(K) \leq g_0(K) \leq \cdots \leq g_n(K)$, and that $g_n(K) = g_n(J)$ if K and J are concordant. Also, K is slice if and only if $g_n(K) = 0$ for some $n \geq -1$. Of course, the n^{th} -order genus of a knot may be considered in either the smooth or topological category.

We prove that the n^{th} -order genus distinguishes knots in \mathcal{G}_{n+2} that are not distinguished by the slice genus. That is, each of our higher-order genera is a refinement of the notion of slice genus.

Theorem 4.10. *For any $n \geq 1$, there is a fixed g and a knot in \mathcal{G}_{n+2} with slice genus bounded above by g and arbitrarily high n^{th} -order genus. Furthermore, this knot has infinite order in $\mathcal{G}_{n+2}/\mathcal{F}_{n.5}$.*

Corollary 4.12. *For any $n \geq 1$, there are infinitely many knots that lie in \mathcal{G}_{n+2} whose slice genera are equal but whose n^{th} -order genera are distinct.*

We can improve Corollary 4.12 in the following sense. In Chapter 5, we prove that g_n is a finer measure than each of the g_i , for $-1 \leq i \leq n-1$. Thus, the n^{th} -order genus gives more information about knots in \mathcal{G}_{n+2} than all of the lower-order genera.

Murasugi proved [Mur, Theorem 9.1] that the ordinary signature of a knot is a lower bound for the slice genus of that knot (henceforth “Murasugi’s inequality”). Gilmer later proved [Gil, Theorem 1] that the sum of certain Casson-Gordon invariants and the ordinary signature bounds the slice genus from below (henceforth “Gilmer’s inequality”). Cochran, Orr, and Teichner first used L^2 -signatures to study knots. These L^2 -signatures are higher-order analogues of the Casson-Gordon invariants. First, we define higher-order analogues of slice genus, and to any (n) -solvable knot we assign a set of real numbers, called the n^{th} -order signatures (see Chapter 3). This begs the question of whether there is a higher-order analogue of Murasugi’s inequality. Our primary tool is the desired higher-order analogue.

Theorem 4.5. *If $K \in \mathcal{G}_{n+2}$, there is an n^{th} -order signature of K that gives a lower bound for the n^{th} -order genus of K .*

We are not the first to utilize L^2 -signatures in the study of genus-like invariants. Cha used metabelian L^2 -signatures to obtain new lower bounds on the minimal genus of embedded surfaces representing a given 2-dimensional homology class in certain 4-manifolds [Cha]. An application of Cha’s methods was to find bounds for the slice genus of knots [Cha, Proposition 5.1]. Our Theorem 4.5 uses the L^2 -signatures to obtain lower bounds for the higher-order genera. While Cha obtained obstructions to slice genus, we obtain higher-order obstructions to the higher-order genera. It seems

that the only (classical) sliceness obstruction our higher-order genera give is that if one of the higher-order genera of a knot is positive, then that knot cannot be slice. However, a knot having large higher-order genera does not in general obstruct the knot from having a small (but positive) slice genus.

We should note that our higher-order signatures give a lower bound on the topological higher-order genera and often fail to be accurate in the smooth category. Consequently, we choose to work in the topological category, except for Section 4.1, which contains examples in the smooth category.

4.1 Concrete examples in the smooth category

In this section we work in the smooth category. The purpose of this section is to construct non-slice knots that bound Groves of a fixed height. We compute the higher-order genera in these examples and conclude that for any positive integers n and m , there is a knot whose smooth n^{th} -order genus is equal to m . The computations do not make use of our n^{th} -order signatures.

Let K denote any knot with non-negative maximal Thurston-Bennequin number. For example, if K is the right-handed trefoil, then $TB(K) = 1$. Let $D(K)$ denote the positively-clasped, untwisted Whitehead double of K as depicted in Figure 4.1. For $i \geq 1$, let $D^i(K) = D(D^{i-1}(K))$ denote the i^{th} iterated Whitehead double of K . By Livingston [Liv], we know that $TB(K) \geq 0$ implies that the Ozsváth-Szabó τ -invariant is nontrivial, i.e. $\tau(D^i(K)) = 1$. It follows that $D^i(K)$ is not smoothly slice for all $i \geq 1$. It should be noted that earlier work of Lee Rudolph implies that

$D^i(K)$ is not slice for all $i \geq 1$ if K is the right-handed trefoil [Rud].

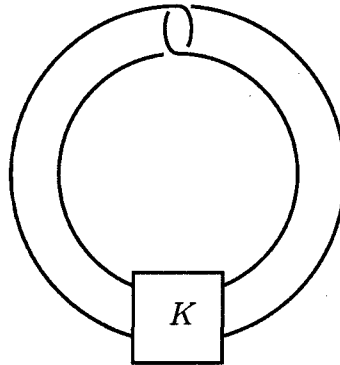


Figure 4.1: $D(K)$: the positively-clasped, untwisted Whitehead double of K

We describe a Grope of height 2 in $S^3 \times I$ bounded by $D(K)$. The standard Seifert surface for $D(K)$ has a symplectic basis of curves, each of which inherits the zero framing from this surface. This basis is pictured in Figure 4.2. Let α denote the basis curve that “goes over the bridge” of this Seifert surface, and let β denote the other curve. Pull α slightly out of the page so that the intersection point with β is removed. Observe that the link $\alpha^+ \cup \beta$ is two parallel copies of K . Now push these two curves down in the I direction and glue parallel Seifert surfaces for K . The Seifert surface for $D(K)$ together with the pushing annuli and Seifert surfaces for K comprise a height 2 Grope for $D(K)$ in $S^3 \times I$. The genus of the first stage of this Grope is 1. Since $1 = \tau(D(K)) \leq g_{-1}(D(K)) \leq g_0(D(K))$ and $g_0(D(K)) \leq 1$ by construction, we have $g_0(D(K)) = 1$.

We can iterate this procedure to build a Grope of height $n + 1$ in $S^3 \times I$ bounded by $D^n(K)$, and the first stage of this Grope has genus 1. As before, we have

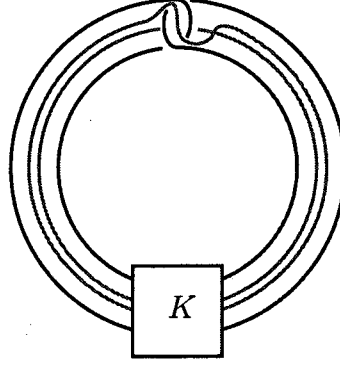


Figure 4.2: A basis of untwisted curves for the Seifert surface of $D(K)$

$1 \leq \tau(D^n(K)) \leq g_{-1}(D^n(K)) \leq g_0(D^n(K)) \leq \cdots \leq g_{n-1}(D^n(K)) \leq 1$, whence $g_{n-1}(D^n(K)) = 1$.

Since $\tau : \mathcal{C} \rightarrow \mathbb{Z}$ is a homomorphism, we conclude that $g_{n-1}(\#_m D^n(K)) \geq \tau(\#_m D^n(K)) = m \cdot \tau(D^n(K)) = m$ and $g_{n-1}(\#_m D^n(K)) \leq m$ by construction.

To summarize, we have the following theorem.

Theorem 4.2. *For any $n \geq 0$ and $m \geq 1$, there is a knot $K \in \mathcal{G}_{n+2}^{smooth}$ of infinite order, and $g_n(K) = m$.*

Remark 4.3. Since the Alexander polynomial of $D(K)$ is trivial, it can be shown that $D(K)$ is smoothly (n) -solvable for all n . However, whether $D(K) \in \mathcal{G}_{n+2}^{smooth}$ for all n is still an open question.

4.2 Lower bounds on higher-order genera

We now turn to our higher-order signatures as tools for estimating the higher-order genera. While the higher-order signatures are not explicitly computable, we demon-

strate how to ensure that all higher-order signatures are large enough to guarantee that the higher-order genera are large.

Lemma 4.4. *Let $K \in \mathcal{F}_n$ and W be an (n) -solution for M_K . Then the n^{th} -order signature of K associated to W satisfies $|\rho(M_K, \phi)| \leq \beta_2(W)$.*

Proof. Let $\phi : \pi_1(M_K) \xrightarrow{i_*} \pi_1(W) \xrightarrow{\pi} \pi_1(W)/\pi_1(W)_r^{(n+1)}$. By the definition of an (n) -solution, the ordinary intersection form of W is a direct sum of hyperbolics, implying that the ordinary signature of W is zero. Since ϕ factors through $\pi_1(W)$, we have that

$$\rho(M_K, \phi) = \sigma^{(2)}(W, \pi) - \sigma(W) = \sigma^{(2)}(W, \pi)$$

Cha has shown that $|\sigma^{(2)}(W, \pi)| \leq \beta_2(W)$ [Cha, Lemma 2.7]. □

That the homomorphism $\phi : \pi_1(M_K) \rightarrow \pi_1(W)/\pi_1(W)_r^{(n+1)}$ factors through $\pi_1(W)$ of bounding 4-manifold W is crucial. Our philosophy differs from Cha's [Cha] in that we assume our homomorphisms factor through bounding 4-manifolds (cf. Definition 3.1), whereas Cha takes a homomorphism $\pi_1(M_K) \rightarrow \Gamma$ and tries to extend it to a bounding 4-manifold. In particular, Cha finds a homomorphism $\phi_\sigma : \pi_1(M_K) \rightarrow \mathbb{Z}$ that factors through a certain bounding 4-manifold, and the von Neumann ρ -invariant associated to this homomorphism satisfies $|\rho(M_K, \phi_\sigma)| \leq 4 g_{-1}(K)$, where $g_{-1}(K)$ is the slice genus of K [Cha, Theorem 1.1 and Proposition 1.2]. We, however, consider many homomorphisms that we assume extend to bounding 4-manifolds, and we show that (at least) one of the associated ρ -invariants satisfies $|\rho| \leq 4 g_n(K)$, where $g_n(K)$ is one of the higher-order genera of K .

Theorem 4.5. *If $K \in \mathcal{G}_{n+2}$, one of the n^{th} -order signatures $\rho \in \mathfrak{S}^n(K)$ satisfies $|\rho| \leq 4g_n(K)$.*

Proof. Let Σ be the first stage of a Grope of height $n + 2$ that realizes $g_n(K)$, i.e. $g(\Sigma) = g_n(K)$. Cochran, Orr, and Teichner construct an (n) -solution W by surgering Σ , and $\beta_2(W) = 4g(\Sigma) = 4g_n(K)$ [COT1, Theorem 8.11]. The conclusion follows from Lemma 4.4. \square

Remark 4.6. Theorem 4.5 may be thought of as a higher-order analogue of Murasugi's inequality [Mur, Theorem 9.1]. Unlike the subsequent inequalities of Gilmer [Gil, Theorem 1] and Cha [Cha, Proposition 5.1], our result gives higher-order obstructions to the higher-order genera.

Corollary 4.7. *If K is a slice knot, then for any n , one of the n^{th} -order signatures of K vanishes.*

Proposition 4.8. *Suppose K is (n) -solvable. If K is $(n.5)$ -solvable, then one of the n^{th} -order signatures of K vanishes.*

Proof. Let W be an $(n.5)$ -solution for K . It follows from [COT1, Theorem 4.2] that the n^{th} -order signature of K associated to W vanishes. \square

Remark 4.9. The conclusion holds even if K is assumed to be merely rationally $(n.5)$ -solvable [COT1, Definition 4.1].

If the Alexander polynomial of a knot is trivial, then the knot is topologically slice [FQ]. In particular, Alexander polynomial one knots are (n) -solvable for all n . Consequently, the n^{th} -order signatures of an Alexander polynomial one knot are

all equal to the classical signature, namely zero. As the n^{th} -order signatures are topological invariants, they will not give accurate bounds for the smooth higher-order genera. For example, the knots constructed in Section 4.1 had trivial Alexander polynomial but large smooth n^{th} -order genera.

Theorem 4.10. *For any $n \geq 1$, there is a fixed g and a knot in \mathcal{G}_{n+2} with slice genus bounded above g and arbitrarily high n^{th} -order genus. Furthermore, this knot has infinite order in $\mathcal{G}_{n+2}/\mathcal{F}_{n.5}$.*

Remark 4.11. The statement of Theorem 4.10 seems to be false for $n = 0$. For example, if $K \in \mathcal{G}_2$, one can construct a Grope of height 2 bounded by K whose first stage has genus equal to the Seifert genus of K . See [COT1, Remark 8.14] for a discussion.

Proof. Let K be an (n) -solvable knot constructed in the proof of Theorem 3.3 with large n^{th} -order signatures. Cochran and Teichner proved that $K \in \mathcal{G}_{n+2}$ [CT, Theorem 3.8], so $g_n(K)$ is defined. Since all of the n^{th} -order signatures for K are large, Theorem 4.5 implies that $g_n(K)$ will be large.

Recall from the proof of Theorem 3.3 that K was built by infection on a ribbon knot R . By [CT, Theorem 4.3], there is a collection of unknotted curves η_i , $1 \leq i \leq j$, in $S^3 - R$ with $[\eta_i] \in \pi_1(M_R)^{(n)}$ and for any (n) -solution V of M_R , some $i_*(\eta_k) \notin \pi_1(V)^{(n+1)}$. Since the η_i bound disjoint discs in S^3 , we can take a Seifert surface for R and tube around the η_i so that the tubes are disjoint. We are left with a Seifert surface for R which the η_i do not intersect. The knot K will have genus bounded above by the genus of our tubed surface for R .

We should note here that since $0 \notin \mathfrak{S}^n(K)$, $K \notin \mathcal{F}_{n.5}$ (by Proposition 4.8). [CT, Theorem 4.2] establishes that K has infinite order in $\mathcal{G}_{n+2}/\mathcal{F}_{n.5}$. \square

Corollary 4.12. *Given any $n \geq 1$, there exist infinitely many knots in \mathcal{G}_{n+2} whose slice genus agree but whose n^{th} -order genera are distinct.*

Proof. By Theorem 4.10, there is a positive integer g and a sequence $\{K_i\}_{i=1}^{\infty}$ of knots in \mathcal{G}_{n+2} with $g_{-1}(K_i) \leq g$ and $g_n(K_i) < g_n(K_{i+1})$ for all $i \geq 1$. Since the set $\{g_{-1}(K_i)\}$ is a finite set, we can pass to a subsequence of knots with the same slice genera but different n^{th} -order genera. \square

Remark 4.13. We can improve the statement of Corollary 4.12 to say that for each $n \geq 1$, there are infinitely many knots in \mathcal{G}_{n+2} with identical i^{th} -order genera for $i \leq n-1$ and distinct n^{th} -order genera. We will state and prove this as Theorem 5.4

Example 4.14. We provide a concrete family of examples of knots $\{L_m\}_{m=1}^{\infty}$ in \mathcal{G}_3 with slice genus bounded above by 3 and $g_1(L_m) < g_1(L_{m+1})$ for all m . Our family is inspired by Cochran, Harvey, and Leidy's family J_n (cf. [CHL4]). Note, this family is not to be confused with the J_m from the proof of Theorem 3.3.

Cochran, Harvey, and Leidy defined their knots by infecting along the curves α and β in Figure 4.3. We cannot use these curves for the purpose of constructing knots bounding Gropes because the two punctured tori bounded by α and β intersect. As per [CT, Lemma 3.9], we find curves α' and β' that are homotopic to α and β , respectively, and that bound disjoint height 1 Gropes in $S^3 - R$. Since these curves are homotopic, the n^{th} -order signatures will not distinguish our examples from the examples of [CHL4]. However, our examples are probably not concordant to theirs.

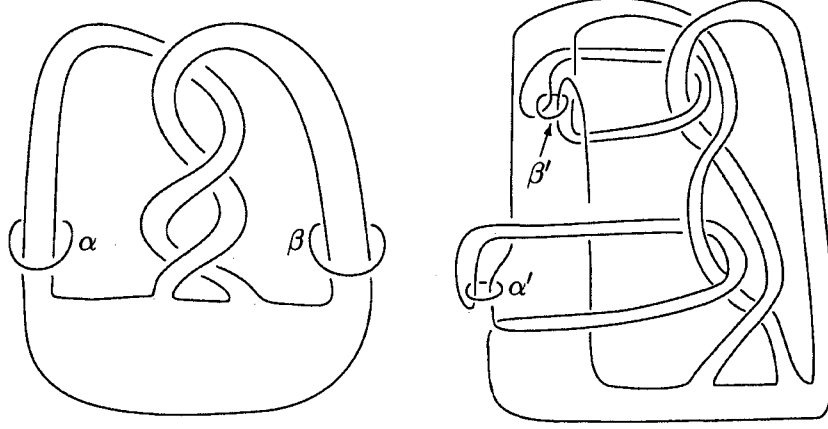


Figure 4.3: The infection curves α and β , and homotopic infection curves α' and β'

Now, let J be the knot from [CT] and let $J_m = \#_m J$ as in the proof of Theorem 3.3. J_m no longer refers to the knots from [CHL4]. Let L_m be infection on $R = 9_{46}$ along α' and β' by J_m . We chose α' and β' so that they bound disjointly embedded punctured tori in the complement of R , so by [CT] the knots L_m will bound Groves of height 3 in D^4 . Since α' and β' lie off of a genus 3 Seifert surface for R , L_m will have slice genus less than or equal to three. It is a consequence of Proposition 5.2 that L_m has slice genus equal to one.

Let V be a (1)-solution for $M = M_{L_m}$. Let $\pi = \pi_1(V)$. Since $H_1(V) \cong \mathbb{Z}$ is torsion-free, we conclude $H_1(V) \cong \pi/\pi^1 \cong \pi/\pi_r^1 \cong \mathbb{Z}$. Let $\phi : \pi_1(M) \xrightarrow{i_*} \pi \rightarrow \pi/\pi_r^1$. Since $i_* : H_1(M) \xrightarrow{\cong} H_1(V) \cong \pi/\pi_r^1$, we see that $\phi : \pi_1(M) \rightarrow H_1(M) \xrightarrow{i_* \cong} H_1(V)$. For emphasis, let $H_1(M; \mathbb{Q}[s, s^{-1}])$ denote the first homology of the infinite cyclic cover of M as a $\mathbb{Q}[s, s^{-1}]$ -module, where $H_1(M) = \langle s \rangle$, and let $H_1(M; \mathbb{Q}[t, t^{-1}])$ denote the first homology induced by the coefficient system $\phi : \pi_1(M) \rightarrow \pi/\pi_r^1$. The curves α and β generate $H_1(M; \mathbb{Q}[s, s^{-1}])$, and since α' and β' are homotopic to these

generators, α' and β' also generate $H_1(M; \mathbb{Q}[s, s^{-1}])$. Since the coefficient system ϕ is $\pi_1(M) \rightarrow H_1(M)$ followed by an isomorphism, α' and β' generate $H_1(M; \mathbb{Q}[t, t^{-1}])$.

Cochran, Orr, and Teichner proved that the coefficient system ϕ induces a hyperbolic bilinear form $\mathcal{B}\ell(\cdot, \cdot)$ defined on $H_1(M; \mathbb{Q}[t, t^{-1}])$ [COT1, Theorem 2.13] and that

$$\mathfrak{k} := \ker\{i_* : H_1(M; \mathbb{Q}[t, t^{-1}]) \rightarrow H_1(V; \mathbb{Q}[t, t^{-1}])\}$$

satisfies $\mathfrak{k} = \mathfrak{k}^\perp$ with respect to this form [COT1, Theorem 4.4]. Thus, $\mathcal{B}\ell(\alpha', \beta')$ is nonzero, and hence one of α' and β' is not in \mathfrak{k} . By the bilinearity of $\mathcal{B}\ell$, all integer multiples of α' or β' are not in \mathfrak{k} . Recall that $H_1(V; \mathbb{Q}[t, t^{-1}])$ is the first homology of the infinite-cyclic cover \tilde{V} of V , viewed as a $\mathbb{Q}[t, t^{-1}]$ -module, and $\pi_1(\tilde{V}) = \pi_1(V)^{(1)}$. If α' were to map into $\pi_1(V)^{(2)}$, then α' would map to zero in $H_1(V; \mathbb{Q}[t, t^{-1}])$. Since no multiple of α' (or of β') lie in \mathfrak{k} , we conclude that α' or β' does not map into $\pi_1(V)_r^{(2)}$. As in Theorem 4.10, we have the following relationship between the ρ -invariants:

$$\rho(M_R, \phi_R) - \rho(M, \phi) = \epsilon_{\alpha'} \rho_0(J_m) + \epsilon_{\beta'} \rho_0(J_m)$$

Since one of α' and β' does not map into $\pi_1(V)_r^{(2)}$, one of $\epsilon_{\alpha'}$ or $\epsilon_{\beta'}$ is one, as discussed in the proof of Theorem 3.3. By choosing m sufficiently large, the number $|\rho(M, \phi)|$ can be made arbitrarily large. Since V was an arbitrary (1)-solution, we have $g_1(L_m)$ is arbitrarily large by Theorem 4.5. Note, one may need to take a subsequence of the constructed L_m to get a family with strictly increasing g_1 .

Chapter 5

Independence of the higher-order genera

We saw in Chapter 4 that for each $n \geq 1$, the n^{th} -order genus is independent of the slice genus. In this chapter, we prove the stronger result that for each $n \geq 1$, the n^{th} -order genus is independent of the $(n - 1)^{\text{th}}$ -order genus. In fact, the n^{th} -order genus is independent of all of the lower-order genera. This result is our Theorem 5.4.

An **annular grope of height n** is a grope of height n that has an extra boundary component on its first stage. We say that the two boundary components of an annular grope cobound an annular grope. Two knots K_0 and K_1 are **height n Grope concordant** if they cobound a height n annular Grope G in $S^3 \times [0, 1]$ such that $G \cap (S^3 \times \{i\}) = K_i$ for $i = 0, 1$. For example, if K is height n Grope concordant to a slice knot, then $K \in \mathcal{G}_n$. The capital “G” in “annular Grope” indicates the untwisted framing.

Let G be a height n Grope in $S^3 \times [0, 1]$ bounded by a knot $K \subset S^3 \times \{0\}$. The

union of sets of basis curves for each n^{th} -stage surface of G is called a **set of tips** for G . A **capped Grope** is the union of a Grope and a disjoint union of discs where the boundaries of the discs form a full set of tips of the Grope. The interiors of these discs must not intersect the Grope except perhaps in the interior of the first stage of the Grope. These discs are called the “caps” of the Grope. In the category of capped gropes, one can speak of “height n capped Grope concordance.”

Recall the knots $L_m \in \mathcal{G}_3$ from Example 4.14. L_m was obtained by infecting 9_{46} along the curves α' and β' by the knot $J_m = \#_m J$, where J is the knot in Figure 3.2. As proven in [CT], J is height 2 capped Grope concordant to the unknot.

Definition 5.1. Suppose K bounds a height $n + 2$ capped Grope in D^4 . We define the n^{th} -order capped genus of K , denoted $g_n^c(K)$, to be the smallest genus of the first stages of all height $n + 2$ capped Gropes bounded by K .

To prove the main result of this chapter, we must construct a height 2 capped Grope bounded by L_m .

Proposition 5.2. *Each of the knots L_m from Example 4.14 bounds a height 2 capped Grope. Furthermore, $g_0^c(L_m) = 1$.*

Remark 5.3. The key property of the L_m is that their 0^{th} -order capped genera are independent of m . Recall, however, that $g_1(L_m)$ grows large as $m \rightarrow \infty$.

Proof of Proposition 5.2. There is a genus 3 Seifert surface, F , for L_m that admits a symplectic basis that is depicted in Figure 5.1. Each curve in this symplectic basis has self-linking zero, and each of them bounds a surface in D^4 that admits caps which

intersect F . The rough idea is to glue these capped surfaces to F , resulting in a height 2 capped Grope bounded by L_m .

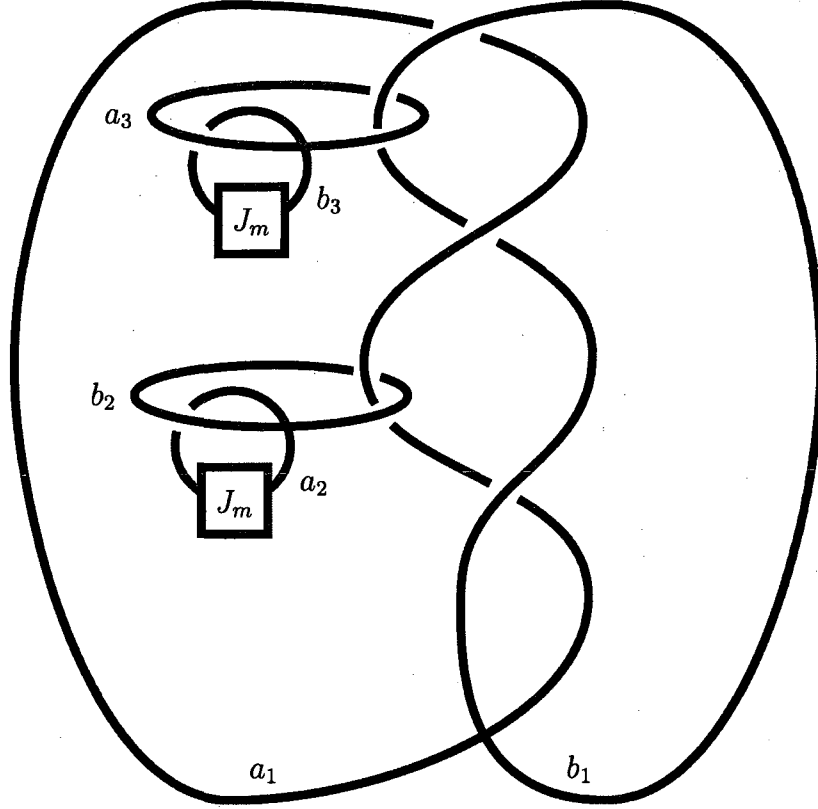


Figure 5.1: A symplectic basis for F

Each of the curves a_1, b_1, b_2 , and a_3 is unknotted. We will think of F as lying in $S^3 \times \{1\}$, and we will build the desired Grope in $S^3 \times [0, 1] \cup D^4$. The first stage of this Grope will be $L_m \times [0, 1] \cup F'$, where F' is the genus one surface obtained by removing the curves a_1 and a_3 from F and gluing on two copies of slice discs for these two curves. Before worrying about F' , let us build second-stage surfaces for a_1, b_1, b_2 , and a_3 that do not intersect F except in these curves. Let A_1, B_1, B_2 , and A_3 be properly embedded discs in D^4 bounded by a_1, b_1, b_2 , and a_3 , respectively, such that

the following hold:

- A_3 and B_1 intersect once in their interiors,
- A_1 and B_2 intersect once in their interiors, and
- A_1 and B_1 intersect once in their interiors and once on their boundaries,
- A_1 and A_3 are disjoint, and
- B_1 and B_2 are disjoint.

Each of the basis curves for F has a normal torus. For example, the normal torus T_{b_1} is the 2-torus that is the restriction of the normal bundle of F to the curve b'_1 , where b'_1 is a push-off of b_1 that still lies in F . One may identify $T_{b_1} \equiv b'_1 \times \mu$, where μ is the meridian of F . One checks that the T_{b_i} are pairwise disjoint, and $T_{b_i} \cap A_j$ is a point if $i = j$ and empty otherwise. Furthermore, $T_{b_i} \cap B_j$ is empty for all i, j .

We now describe a procedure for eliminating the intersection points of B_j with A_i . Pick a path p in A_i joining an interior intersection point of A_i and B_j to the intersection point of A_i with T_{b_i} . We may remove neighborhoods in B_j and T_{b_i} of these intersection points and join the two circles with a tube that runs along p . We will call this procedure “tubing B_j into T_{b_i} ” and we will denote the result as $B_j \circ T_{b_i}$ (we will not bother by incorporating the path p into this notation, as the choice of path is unimportant). This procedure reduces $|\mathring{A}_i \cap \mathring{B}_j|$ by one and increases the genus of B_j by one.

Let $B'_1 = (B_1 \circ T_{b_1}) \circ T_{b_3}$ and $B'_2 = B_2 \circ T_{b_1}$. To guarantee B'_1 and B'_2 are disjoint, one can choose a tube for B'_2 that “goes inside” the tube used in the operation $B_1 \circ T_{b_1}$,

and tube into a smaller copy of T_{b_1} . A schematic diagram of this operation is depicted in Figure 5.2.

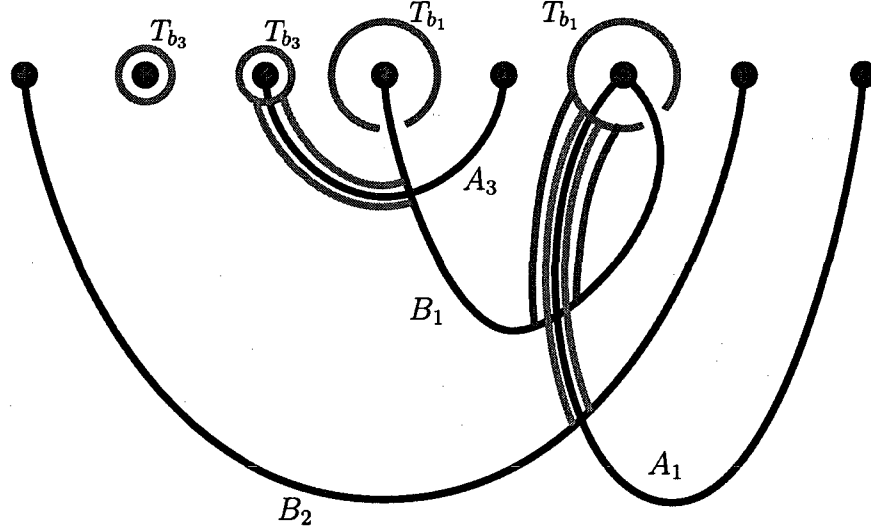


Figure 5.2: Tubing the discs into the normal tori

We may surger F using the discs A_1 and A_3 . The result is a surface $F' \subset D^4$ of genus one. The symplectic basis $\{a_2, b_2\}$ for F' lies in $S^3 \times \{1\}$. Since the second-stage surface B'_2 for b_2 was disjoint from $F \cup A_1 \cup A_2$, B'_2 is disjoint from the surgered surface F' . We may choose a symplectic basis $\{b'_1 \times \vec{v}, x \times \mu\}$ so that \vec{v} is the normal direction of F pointing “up” in the $[0, 1]$ direction of $S^3 \times [0, 1]$, and x is a point on b'_1 . Thus, we may view $b'_1 \times \vec{v}$ as a copy of b_1 lying in $S^3 \times \{1 - \epsilon\}$ for a sufficiently small ϵ . In $S^3 \times \{1 - \epsilon\}$, we may glue a cap (disc) to $b_1 \times \vec{v}$ that intersects $L_m \times \{1 - \epsilon\}$. Thus, this cap hits the first stage $L_m \times [0, 1] \cup F'$ transversely. We may also cap off the curve $x \times \mu$ so that it hits F' transversely.

Float the curve a_2 into the level $S^3 \times \{1 - \frac{\epsilon}{2}\}$, below $b_1 \times \vec{v}$. We must caution the reader that part of B'_2 lies in $S^3 \times \{1 - \frac{\epsilon}{2}\}$. This part of B'_2 is two parallel copies of the

curve b_1 , which by Figure 5.1, is split off from a_2 . We claim that in this $S^3 \times \{1 - \frac{\epsilon}{2}\}$, (1) we may place a Seifert surface S for $a_2 = J_m$ that is disjoint from $L_m \times \{1 - \frac{\epsilon}{2}\}$, (2) there exist (intersecting) caps for S that hang further up into $S^3 \times [1 - \frac{2\epsilon}{3}, 1 - \frac{\epsilon}{2}]$, and (3) we may resolve these cap-cap intersections. Part (1) is nothing new, and it follows immediately since L_m is a satellite of J_m (one must note that a_2 is isotopic in $S^3 - L_m$ to a preferred longitude of the companion torus for J_m). The idea behind part (1) can be seen in the Whitehead double example in Section 4.1. To prove part (2), we may assume without loss of generality that $m = 1$. A projection of $J = J_1$ is shown in Figure 5.3. There is an obvious genus one “ribbon Seifert surface” for J , and one may resolve the ribbon singularities to obtain a genus 5 Seifert surface S for J . One may check that this surface S admits a symplectic basis of curves, each of which has self-linking zero and is unknotted. Thus, we may hang caps for this basis up into $S^3 \times [1 - \frac{2\epsilon}{3}, 1 - \frac{\epsilon}{2}]$ that miss the rest of the Grope, but some of the caps will intersect transversely. To prove part (3), we use the finger move of [FQ] to exchange these cap-cap intersections for intersections of the caps with F' , as desired.

The desired capped Grope H can be described as follows:

- first-stage surface: $L_m \times [0, 1] \cup F'$,
- second-stage surfaces: S and B'_2 ,
- caps as described above.

Since L_m is not slice, $g_0^c(L_m) \neq 0$. The genus of the first stage of H is one. \square

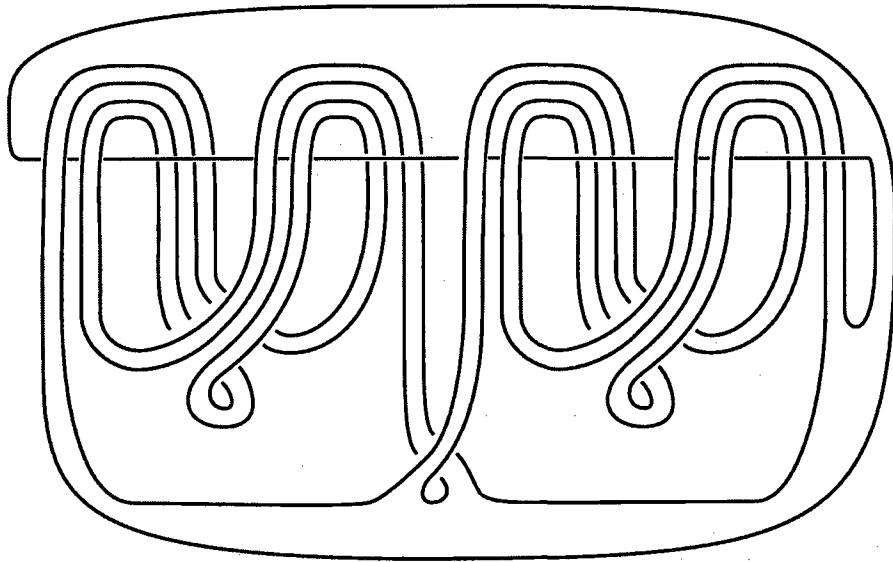


Figure 5.3: A projection of J

5.1 Extending the family

We recall that the knots L_m lie in \mathcal{G}_3 and have increasing 1st-order genera. By iterating the infection process, we will produce for each $n \geq 1$ a family L_m^n of knots in \mathcal{G}_{n+2} . A subfamily of these knots will have the properties listed in Theorem 5.4.

For $n = 1$, we define L_m^1 to be the knot L_m . Suppose we have defined L_m^n . We define L_m^{n+1} to be the result obtained by infecting 9_{46} along α' and β' by L_m^n (use the same infecting knot for each infection curve). We proved in [Hor1] that $L_m^n \in \mathcal{G}_{n+2}$ for each n, m .

Cochran, Harvey, and Leidy gave an alternate description of L_m^n . Let $R^1 = 9_{46}$. Let R^{n+1} be the result obtained by infecting R^1 along α' and β' by R^n . One can show that each iteration of this infection produces two “ghosts” of α' and two “ghosts” of β' . In other words, R^n will have 2^{n+1} “ghosts” (2^n of α' and 2^n of β'). One obtains

L_m^n by infecting R^n along these 2^{n+1} curves by $\#_m J$. These ghost infection curves lie in the n^{th} term of the derived series of $\pi_1(S^3 - R^n)$. In [CHL1, Theorem 4.11], Cochran, Harvey, and Leidy proved that at least one of these ghosts survives under the map

$$\pi_1(S^3 - R^n) \rightarrow \pi_1(M_{R^n}) \rightarrow \pi_1(V) \twoheadrightarrow \pi_1(V)/\pi_1(V)_r^{(n+1)}$$

where V is any (n) -solution for M_{R^n} . Thus, by the arguments in the proofs of Theorems 3.3 and 4.10, the n^{th} -order genera of L_m^n grow as $m \rightarrow \infty$.

5.2 Proof of Theorem 5.4

Theorem 5.4. *For $n \geq 1$, there exists a knot in \mathcal{G}_{n+2} whose n^{th} -order genus is arbitrarily larger than all of its lower-order genera, up to order $n - 1$. Furthermore, there is an infinite family of knots in \mathcal{G}_{n+2} all of whose members are distinguished by their n^{th} -order genera but with identical i^{th} -order genera for $-1 \leq i \leq n - 1$.*

Proof. Fix an $n \geq 1$. We will show that all of the L_m^n satisfy $g_{n-1}(L_m^n) \leq 4^{n-1}$. However, by the discussion in Section 5.1, $g_n(L_m^n)$ tends to infinity as $m \rightarrow \infty$. Since $g_i(K) \leq g_{i+1}(K)$ for all i and any knot K for which g_{i+1} is defined, we see that for $i = -1, \dots, n - 1$, $g_i(L_m^n) \leq 4^{n-1}$. Since this bound is independent of m , there is a subsequence $L_{m_j}^n$ of the knots L_m^n with the properties that

- for each $i = -1, \dots, n - 1$, $g_i(L_{m_j}^n) = g_i(L_{m_{j'}}^n)$ for all j, j' , and
- $g_n(L_{m_1}^n) < \dots < g_n(L_{m_k}^n) < g_n(L_{m_{k+1}}^n) < \dots$

Thus, the proof will be complete once we verify $g_{n-1}(L_m^n) \leq 4^{n-1}$.

We construct Grope according to [Hor1]. Recall the notation of Section 3.2. Proposition 3.2 of [Hor1] tells us that “infection by height n Grope concordant knots results in height n Grope concordant knots.” More specifically, if R is a seed knot with infection curve α , and if T and T' are height n Grope concordant knots, then $R(\alpha, T)$ and $R(\alpha, T')$ are height n Grope concordant knots. One can build a Grope concordance between $R(\alpha, T)$ and $R(\alpha, T')$ by gluing slit Grope concordances (from T to T') along the strands of R encircled by the infection curve α .

Presently, we have described L_m^2 as the result of infecting 9_{46} along α' and β' by L_m^1 . Recall from Proposition 5.2 that L_m^1 is height 2 capped Grope concordant to the unknot, and this Grope concordance H has genus one. Thus, by the preceding paragraph, L_m^2 is height 2 Grope concordant to 9_{46} (viewed as the result of infecting 9_{46} along α' and β' by the unknot). Each of the infection curves encircles two strands of 9_{46} , so according to [Hor1, Proposition 3.2], there is a height 2 Grope concordance between L_m^2 and 9_{46} that is obtained by gluing $2 \cdot 2 = 4$ slit copies of H together. Attaching a ribbon disc to 9_{46} yields a height 2 Grope G for L_m^2 of genus 4. Recall that H was a capped Grope concordance. We may view the caps as hitting H in meridians of H 's boundary knot. In the process of infection, the meridian of the infecting knot is identified with the longitude of the infection curve. Thus, the tips of G are isotopic to parallel copies of α' and β' . Recall that α' and β' bound disjoint tori in the complement of 9_{46} , and each of these tori is capped (the caps hit 9_{46}); an explicit description, including a diagram, of this was given in [Hor1, Section 3.2]. Gluing on many parallel copies of these capped tori (which may be done in D^4) to the Grope G increases the height by one. Therefore, G together with the capped tori

forms a height 3 capped Grope of genus 4. We have verified that

$$g_1^c(L_m^2) \leq 4$$

We repeat this procedure ad infinitum. For example, we can repeat this construction to get a height $n + 1$ capped Grope for L_m^n by using four copies of the height n capped Grope concordance between L_m^{n-1} and the unknot. Thus,

$$g_{n-1}^c(L_m^n) \leq 4 g_{n-2}^c(L_m^{n-1})$$

Combining this inequality for all $1 \leq i \leq n - 1$, we see that

$$g_{n-1}(L_m^n) \leq g_{n-1}^c(L_m^n) \leq 4^{n-1}$$

as desired. □

5.3 Application to a geometric structure on the Grope filtration

Let B_n^r denote the subset of all K in \mathcal{G}_{n+2} such that $g_n(K) \leq r$. Since $g_{-1} \leq g_0 \leq \dots \leq g_n$, we see that $B_{-1}^r \supseteq B_0^r \supseteq \dots \supseteq B_n^r$. Theorem 4.10 states that the higher-order genera are finer measures than the slice genus. Furthermore, by Theorem 5.4, the n^{th} -order genus is a finer measure than the lower-order genera, up to order $n - 1$. That is, for some r , these subset containments are proper. Consequently, these higher-order genera provide a further refinement of the Grope filtration of the knot concordance group. That is, after determining how deep a knot lies in the Grope filtration (say in \mathcal{G}_{n+2}), one might try to determine the knot's n^{th} -order genus.

We attempt to complement these comments with the diagram in Figure 5.4. The ambient three-dimensional space represents \mathcal{G}_{n+2} , the plane represents \mathcal{G}_{n+3} , the line represents \mathcal{G}_{n+4} , and the origin represents $\bigcap_{n \geq 0} \mathcal{G}_n$. The corresponding balls have been drawn. The diagram suggests the existence of knots in $B_n^r - B_{n+1}^r$, which was proven in Theorems 4.10 and 5.4 for certain r .

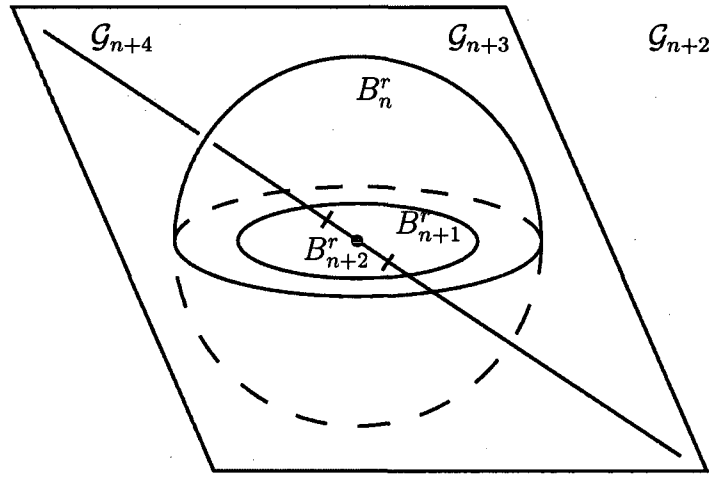


Figure 5.4: The refinement of the Grope filtration by the higher-order genera

Bibliography

- [CG1] Andrew J. Casson and Cameron McA. Gordon. On slice knots in dimension three. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, CA, 1976)*, volume XXXII of *Proceedings of Symposia in Pure Mathematics*, pages 39–53, Providence, RI, 1978. American Mathematical Society.
- [CG2] Andrew J. Casson and Cameron McA. Gordon. Cobordism of classical knots. In *À la recherche de la topologie perdue*, volume 62 of *Progress in Mathematics*, pages 181–199. Birkhäuser Boston, Boston, MA, 1986.
- [CG3] Jeff Cheeger and Mikhael Gromov. Bounds on the von Neumann dimension of L^2 -cohomology and the Gauss-Bonnet theorem for open manifolds. *Journal of Differential Geometry*, 21:1–34, 1985.
- [Cha] Jae Choon Cha. Topological minimal genus and L^2 -signatures. *Algebraic & Geometric Topology*, 8:885–909, 2008.
- [CHL1] Tim Cochran, Shelly Harvey, and Constance Leidy. Knot concordance and Blanchfield duality. Available at <http://arxiv.org/pdf/0705.3987>.

- [CHL2] Tim Cochran, Shelly Harvey, and Constance Leidy. Derivatives of knots and second-order signatures. <http://arxiv.org/pdf/0808.1432>, 2008.
- [CHL3] Tim Cochran, Shelly Harvey, and Constance Leidy. Link concordance and generalized doubling operators. *Algebraic & Geometric Topology*, 8:1593–1646, 2008.
- [CHL4] Tim Cochran, Shelly Harvey, and Constance Leidy. Knot concordance and higher-order Blanchfield duality. *Geometry & Topology*, 13:1419–1482, 2009.
- [Coc] Tim Cochran. Noncommutative knot theory. *Algebraic & Geometric Topology*, 4:347–398, 2004.
- [Coh] Paul M. Cohn. *Free Rings and Their Relations*, volume 19 of *London Mathematical Society Monographs*. Academic Court [Harcourt Brace Janovich Publishers], London, 1985.
- [COT1] Tim Cochran, Kent Orr, and Peter Teichner. Knot concordance, Whitney towers and L^2 -signatures. *Annals of Mathematics*, 157:433–519, 2003.
- [COT2] Tim Cochran, Kent Orr, and Peter Teichner. Structure in the classical knot concordance group. *Commentarii Mathematici Helvetici*, 79:105–123, 2004.
- [Cro] Peter Cromwell. Homogeneous links. *Journal of the London Mathematical Society*, 39(2):535–552, 1989.
- [CT] Tim Cochran and Peter Teichner. Knot concordance and von Neumann ρ -invariants. *Duke Mathematical Journal*, 137(2):337–379, 2007.

- [Cul] Marc Culler. Gridlink 2.0. software, available at <http://www.math.uic.edu/~culler/gridlink>.
- [FQ] Michael Freedman and Frank Quinn. *Topology of 4-Manifolds*. Number 39 in Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1990.
- [Fri] Stefan Friedl. Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants. *Algebraic & Geometric Topology*, 4:893–934, 2004.
- [FT] Michael Freedman and Peter Teichner. 4-manifold topology I: Subexponential groups. *Inventiones Mathematicae*, 122:509–529, 1995.
- [Gil] Patrick Gilmer. On the slice genus of knots. *Inventiones Mathematicae*, 66:191–197, 1982.
- [Har] Shelly Harvey. Homology cobordism invariants and the Cochran-Orr-Teichner filtration of the link concordance group. *Geometry and Topology*, 12:387–430, 2008.
- [Hed1] Matthew Hedden. Knot Floer homology of Whitehead doubles. *Geometry and Topology*, 11:2277–2338, 2007.
- [Hed2] Matthew Hedden. In preparation. 2008.
- [Hor1] Peter Horn. The non-triviality of the Grope filtrations of the knot and link concordance groups. Available at <http://arxiv.org/abs/0804.2661>.

- [Hor2] Peter Horn. The first-order genus of a knot. *Mathematical Proceedings of the Cambridge Philosophical Society*, 146(1):135–149, 2009.
- [Juh] András Juhász. Knot Floer homology and Seifert surfaces. *Algebraic and Geometric Topology*, 8:603–608, 2008.
- [Kim1] Taehee Kim. Filtration of the classical knot concordance group and Casson-Gordon invariants. *Mathematical Proceedings of the Cambridge Philosophical Society*, 137(2):293–306, 2004.
- [Kim2] Taehee Kim. An infinite family of non-concordant knots having the same Seifert form. *Commentarii Mathematici Helvetici*, 80(1):147–306, 2005.
- [Lei] Constance Leidy. Higher-order linking forms for knots. *Commentarii Mathematici Helvetici*, 81(4):755–781, 2006.
- [Let] Carl F. Letsche. An obstruction to slicing knots using the eta invariant. *Mathematical Proceedings of the Cambridge Philosophical Society*, 128(2):301–319, 2000.
- [Liv] Charles Livingston. Computations of the Ozsváth-Szabó knot concordance invariant. *Geometry and Topology*, 8:735–742, 2004.
- [Lüc] Wolfgang Lück. *L^2 -Invariants: Theory and Applications to Geometry and K-Theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2002.

- [Min] Yair Minsky. The classification of punctured-torus groups. *Annals of Mathematics*, 149:559–626, 1999.
- [Mur] Kunio Murasugi. On a certain numerical invariant of link types. *Transactions of the American Mathematical Society*, 117:387–422, 1965.
- [OS1] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Advances in Mathematics*, 186(1):58–116, 2004.
- [OS2] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and genus bounds. *Geometry and Topology*, 8:311–334, February 2004.
- [Pas] Donald S. Passman. *The Algebraic Structure of Group Rings*. Robert E. Krieger Publishing Company, Melbourne, FL, 1985.
- [Ras] Jacob Rasmussen. *Floer homology and knot complements*. PhD thesis, Harvard University, 2003.
- [Rei] Holger Reich. *Group von Neumann Algebras and Related Algebras*. PhD thesis, Georg-August-Universität zu Göttingen, 1998.
- [Rud] Lee Rudolph. Quasipositivity as an obstruction to sliceness. *Bulletin of the American Mathematical Society*, 29(1):51–59, 1993.
- [Sei] Herbert Seifert. Über das Geschlecht von Knoten. *Mathematische Annalen*, 110(1):571–592, 1935.
- [Suz] Shinichi Suzuki. *Introduction to Knot Theory*. Saien-su-sha, (in Japanese), 1991.

- [Whi] Wilbur Whitten. Isotopy types of knot spanning surfaces. *Topology*, 12:373–380, 1973.